University of Vaasa, spring 2014

Introduction to Mathematical Economics, ORMS1030

Exercise 3, week 5 (Jan 27–31, 2014)

R1	ma	10–12	D115	R5	ti	14–16	B209
R2	ma	14–16	D103	R6	to	12–14	C209
R3	ti	08–10	F104	R7	pe	08–10	A201
R4	ti	12–14	F119	R8	pe	10–12	D115

1. The TehFin enterprise produce several different products. One of the production lines has demand $p = 20 - 0.030q$ and cost function $C(q) = 0.02q^2 + 5q + 150$. ($q$ is number of products per month, C is euros per month). Find the optimal production by which the profit is maximal? What is the maximum of the profit?

Positive cash flow:

$$\begin{aligned}
&\text{demand:} & p &= 20 - 0.030q \\
&\text{revenue:} & R(q) = pq &= (20 - 0.030q)q \\
& & &= 20q - 0.030q^2 \\
&\text{marginal revenue:} & MR = R'(q) &= 20 - 0.06q
\end{aligned}$$

Negative cash flow:

$$\begin{aligned}
&\text{cost:} & C(q) &= 0.02q^2 + 5q + 150 \\
&\text{marginal cost:} & MC = C'(q) &= 0.04q + 5
\end{aligned}$$

Optimal production:

$$\begin{aligned}
MC &= MR \\
\Leftrightarrow 0.04q^* + 5 &= 20 - 0.06q^* \\
\Leftrightarrow 0.10q^* &= 15 \\
\Leftrightarrow q^* &= 150
\end{aligned}$$

Maximal profit:

$$\begin{aligned}
P(q^*) &= R(150) - C(150) \\
&= (20 \cdot 150 - 0.030 \cdot 150^2) - (0.02 \cdot 150^2 + 5 \cdot 150 + 150) = 975
\end{aligned}$$

Answer: Optimal monthly production is 150 products, and the profit then is 975 €/month.

2. TehFin owns an factory called "tehas" in the Laihia village. Tehas makes protection relays. Production is now q relays in a week, and TehFin sells the relays by the price of p (€/relay). Demand function is $p(q) = 5 - 0.01q$. The variable production cost is 1.5 €/relay, and the fixed production cost is 230 €/week.
The production capacity of Tehas is 150 relays per week. Tehas can exceed its capacity if the excess of the production is made using overtime work. For the products made in overtime the variable production cost is 1.6 €/relay. If the overtime work is done, then the fixed cost will be 250 €/week. Find the optimal production of Tehas?

Revenue is $R(q) = p(q) \cdot q = (5 - 0.01q)q = 5q - 0.01q^2$. The cost function is non-continuous:

$$\begin{aligned} C(q) &= \begin{cases} 1.5 \cdot q + 230, & \text{if } q \leq 150 \\ 1.5 \cdot 100 + 1.6 \cdot (q - 150) + 250, & \text{if } q > 150 \end{cases}, \\ &= \begin{cases} 1.5 \cdot q + 230, & \text{if } q \leq 150 \\ 1.6 \cdot q + 235, & \text{if } q > 150 \end{cases}. \end{aligned}$$

$$\begin{aligned} \text{Profit:} \quad P(q) &= R(q) - C(q) \\ &= \begin{cases} -0.01q^2 + 3.5 \cdot q - 230, & \text{if } q \leq 150 \ \text{(region I)} \\ -0.01q^2 + 3.4 \cdot q - 235, & \text{if } q > 150 \ \text{(region II)} \end{cases}. \end{aligned}$$

In region I:

$$\begin{aligned} P_I(q) &= -0.01q^2 + 3.5 \cdot q - 230 \\ \Rightarrow P_I'(q) &= -0.02q + 3.5 \quad \longrightarrow P_I' > 0, \text{when } q < 175 \end{aligned}$$

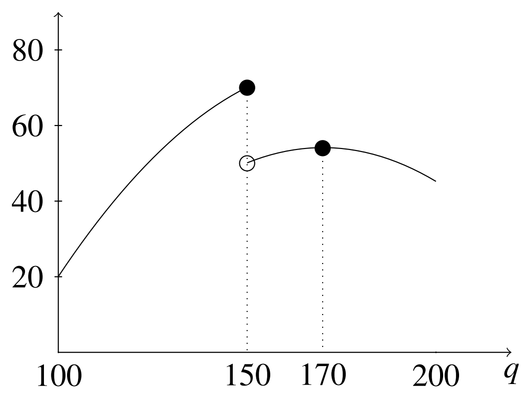
So in the whole region I ($0 \leq q \leq 150$) Profit is increasing. Best choise for production in region I is $q_1 = 150$!
In region II:

$$\begin{aligned} P_{II}(q) &= -0.01q^2 + 3.4 \cdot q - 235 \\ \Rightarrow P_{II}'(q) &= -0.02q + 3.4 \quad \longrightarrow P_{II}' > 0, \text{when } q < 170 \end{aligned}$$

So in the region II ($150 \leq q \leq \infty$) Profit is increasing from 150 to 170, and decreasing fro 170 to ∞. Best choise for production in region II is $q_2 = 170$!
Now we choose the best out of the bests.

$$\begin{aligned} P(150) &= -0.01 \cdot 150^2 + 3.5 \cdot 150 - 230 = 70 \quad \text{Best} \\ P(170) &= -0.01 \cdot 170^2 + 3.4 \cdot 170 - 235 = 54 \end{aligned}$$

Answer: Optimal production is 150 products in week.



3. For a Product A the demand function is $p_A = 20 - 0.2q_A$. For product B the relation between the unit price p_B and the demand q_B is $q_B = 6000/p_B^2$.
a) For both products draw the demand function in the form $p = f(q)$.
b) For both products calculate the price elasticity of demand, when $q_1 = 20$, and $q_2 = 21$.
c) For both products calculate the price elasticity of demand, when $q_1 = 80$, and $q_2 = 81$.

$$\text{price elasticity of demand} = \frac{(q_2 - q_1)}{(p_2 - p_1)} \cdot \frac{p_1}{q_1}, \quad p_1 = f(q_1), p_2 = f(q_2)$$

a)
$$q_B = 6000/p_B^2 \qquad \cdot p_B^2 \;\; : q_B$$
$$p_B^2 = \frac{6000}{q_B} \qquad \longrightarrow p_B = \sqrt{6000/q_B}$$

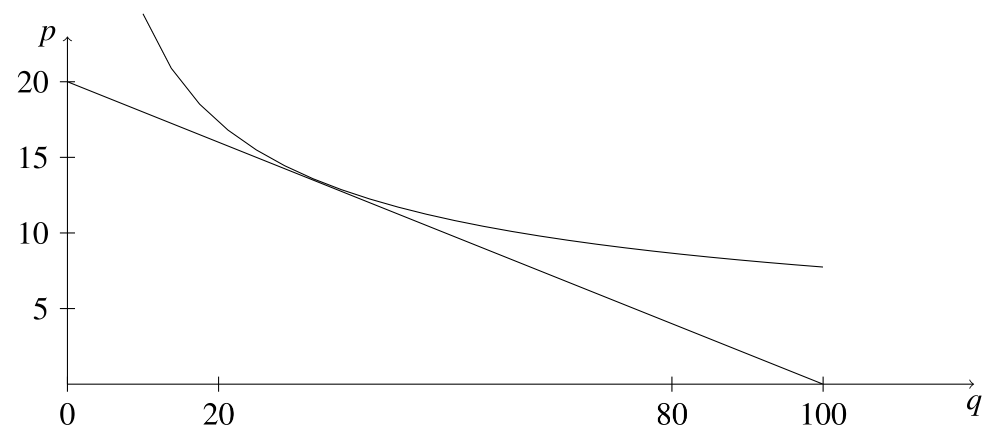
Answer to a) $p_A = 20 - 0.2q_A$, $p_B = \sqrt{6000/q_B}$.

b)
$$\begin{aligned} p_A(20) = 20 - 0.2 \cdot 20 &= 16.00 \\ p_A(21) = 20 - 0.2 \cdot 21 &= 15.80 \\ E_A(20) &= \frac{\Delta q}{\Delta p} \cdot \frac{p_1}{q_1} = \frac{1}{-0.2} \cdot \frac{16}{20} = -4 \end{aligned}$$

$$\begin{aligned} p_B(20) = \sqrt{6000/20} &= 17.3205 \\ p_B(21) = \sqrt{6000/21} &= 16.9031 \\ E_B(20) &= \frac{\Delta q}{\Delta p} \cdot \frac{p_1}{q_1} = \frac{1}{-0.4174} \cdot \frac{17.3205}{20} = -2.075 \end{aligned}$$

b)
$$\begin{aligned} p_A(80) = 20 - 0.2 \cdot 80 &= 4.00 \\ p_A(81) = 20 - 0.2 \cdot 81 &= 3.80 \\ E_A(20) &= \frac{\Delta q}{\Delta p} \cdot \frac{p_1}{q_1} = \frac{1}{-0.2} \cdot \frac{4}{80} = -0.25 \end{aligned}$$

$$\begin{aligned} p_B(80) = \sqrt{6000/80} &= 8.6603 \\ p_B(81) = \sqrt{6000/81} &= 8.6066 \\ E_B(80) &= \frac{\Delta q}{\Delta p} \cdot \frac{p_1}{q_1} = \frac{1}{-0.0537} \cdot \frac{8.6603}{80} = -2.016 \end{aligned}$$



4. The hit product of TehFin is called "Fyrry". Fyrrys price elasticity of demand is $-1{,}75$. The unit price of Fyrry is 12,50€, and Fyrrys demand is 1 200 per month.
a) Find the change of demand if the unit price of Fyrry is decreased by one euro.
b) Find the change of revenue if the unit price of Fyrry is decreased by one euro.
c) The marginal cost of production is $MC = 8{,}00$€. Is it profitable to decrease the unit price by one euro?

a)
$$\begin{aligned} q &= 1\,200\text{kpl/month} \\ p &= 12.50€ \\ \Delta q &= x \\ \Delta p &= -1.00€ \\ \text{elasticity} &= -1.75 \end{aligned}$$

$$\begin{aligned} \frac{\Delta q}{\Delta p}\cdot\frac{p}{q} &= \text{elasticity} \\ \frac{x}{-1.00€}\cdot\frac{12.50€}{1\,200\text{kpl/month}} &= -1.75 \\ x &= \frac{-1.75\cdot(-1.00€)\cdot 1\,200\text{kpl/month}}{12.50€} \\ x &= +168\text{kpl/month} \end{aligned}$$

So the demand increases by 168 products per month, and the new demand is so 1 368 kpl/month. (The percentage change is $-1.75\cdot(-8\%) = +14\%$ and change of numbers is $+0.14\cdot 1200 = 168$.)

b)
$$\begin{aligned} \text{old revenue } = R_1 &= 1200\frac{\text{kpl}}{\text{month}}\cdot 12.50\frac{€}{\text{kpl}} = 15\,000\frac{€}{\text{month}} \\ \text{new revenue } = R_2 &= 1368\frac{\text{kpl}}{\text{month}}\cdot 11.50\frac{€}{\text{kpl}} = 15\,732\frac{€}{\text{month}} \end{aligned}$$

$$\text{change of revenue } = \Delta R = 15\,732\frac{€}{\text{month}} - 15\,000\frac{€}{\text{month}} = 732\frac{€}{\text{month}}$$

c) Production cost increase by

$$\Delta C \approx MC\cdot\Delta q = 8.00\frac{€}{\text{kpl}}\cdot 168\frac{\text{kpl}}{\text{month}} = 1\,344\frac{€}{\text{month}}$$

Answer: a) The demand increase by 168 kpl/month (14%). b) The revenue increase by 732 euro/month (4.9%). c) The production cost increase by 1 344 euro/month. So it is not profitable to decrease the unit selling price by one euro.

5. A big laundry needs monthly 2 500 pots certain special detergent. (2 liter per pot.) Annual holding cost per unit is $h = 0.5€/\text{pot}/\text{year}$. The fixed cost per order is $K = 75€/\text{order}$. The laundry makes orders of 5 000 pots. How much can laundry save by optimizing the order quantity? What is the optimal order quantity? Are the savings remarkable or irrelevant for the big laundry?

$D = 2\,500$pots/month $= 30\,000$pots/year, $K = 75€$, $h = 0.5€$/pot/year.

Optimal way to do things:

$$\begin{aligned} q_0 = \sqrt{\frac{2KD}{h}} &= \sqrt{\frac{2 \cdot 75 \cdot 30\,000}{0.5}}\text{pots} = 3000\text{pots} \\ TC_0 = \frac{KD}{q_0} + h\frac{q_0}{2} &= \frac{75€ \cdot 30\,000\text{pots/year}}{3000\text{pots}} + 0.5\frac{€}{\text{pots} \cdot \text{year}} \cdot \frac{3000\text{pots}}{2} \\ &= 750\frac{€}{\text{year}} + 750\frac{€}{\text{year}} = 1\,500\frac{€}{\text{year}} \end{aligned}$$

The actual way they do it: The laundry makes order every second year!

$$\begin{aligned} TC(q = 5000) = \frac{KD}{5000} + h\frac{5000}{2} &= \frac{75€ \cdot 30\,000\text{pots/year}}{5000\text{pots}} + 0.5\frac{€}{\text{pots} \cdot \text{year}} \cdot \frac{5000\text{pots}}{2} \\ &= 450\frac{€}{\text{year}} + 1\,250\frac{€}{\text{year}} = 1\,700\frac{€}{\text{year}} \end{aligned}$$

There is a possibility to save about 200 euros/year (11.8% of current cost). This is a remarkable percentage. The laundry should change its ordering schedule immediately.

6. The annual demand of ink cartridges is $D = 1600$. The fixed cost per order is $K = 9€$ and the monthly holding cost per unit is 1.5€/month/cartridge.

a) Find the optimal order quantity q_0, and the minimum of the total cost TC_0.

b) The unit purchase price of ink cartridge is 3,50€. The supplier of ink cartridges makes an offer of volume discount, giving 1% discount if order is at least 50 cartridges, and 3% discount if order is at least 100 cartridges. Find the optimal order quantity in the presence of the offer of the volume discount.

$D = 1600$kpl/year, $K = 9€$, $h = 1.5€$/kpl/month = $h = 18€$/kpl/year.

a)

$$\begin{aligned} q_0 &= \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 9 \cdot 1600}{18}} = 40\text{kpl} \\ TC_0 &= \frac{KD}{q_0} + h \cdot \frac{q_0}{2} = \frac{9 \cdot 1600}{40} + 18 \cdot \frac{40}{2} = 360\frac{€}{\text{year}} + 360\frac{€}{\text{year}} = 720\frac{€}{\text{year}} \end{aligned}$$

b)

$$\begin{aligned} TC^*(q) &= \frac{KD}{q} + h \cdot \frac{q}{2} + p \cdot D \\ &= \frac{\text{order cost}}{\text{year}} + \frac{\text{holding cost}}{\text{year}} + \frac{\text{purchase cost}}{\text{year}} \end{aligned}$$

$$\begin{aligned} TC^*(40) &= \frac{9 \cdot 1600}{40} + 18 \cdot \frac{40}{2} + 3.50 \cdot 1600 = 6\,320€/\text{year} \\ TC^*(50) &= \frac{9 \cdot 1600}{50} + 18 \cdot \frac{50}{2} + 0.99 \cdot 3.50 \cdot 1600 = 6282€/\text{year} \quad *** \\ TC^*(100) &= \frac{9 \cdot 1600}{100} + 18 \cdot \frac{100}{2} + 0.97 \cdot 3.50 \cdot 1600 = 6\,476€/\text{year} \end{aligned}$$

So the optimal order quantity is now $q_{opt} = 50$kpl.

Formulas:

Price elasticity of demand:

$$E_{qp} = \frac{\Delta q}{\Delta p} \cdot \frac{p}{q} \quad = \quad \text{price elasticity of demand}$$

$$MR \quad = \quad p\left(1 + \frac{1}{E_{qp}}\right)$$

EOQ:

basic model
$$q_0 = \sqrt{\frac{2KD}{h}}$$
$$TC_0(q) = \frac{KD}{q} + h \cdot \frac{q}{2}$$

slack allowed
$$q_1 = q_0\sqrt{\frac{h+s}{s}}, \quad M_1 = q_0\sqrt{\frac{s}{h+s}},$$
$$TC_1(q) = \frac{KD}{q} + \frac{M^2 h}{2q} + \frac{(q-M)^2 s}{2q}$$

production model
$$q_2 = q_0\sqrt{\frac{r}{r-D}}, \quad M_2 = q_0\sqrt{\frac{r-D}{r}},$$
$$TC_2(q) = \frac{KD}{q} + \frac{hq(r-D)}{2r}$$