

10. harjoitus, viikko 15 (2.4.-6.4.2011)

R1	ma	10-12	D115	R4	to	08-10	D115
R2	ma	14-16	D102	R5	to	14-16	D102
R3	ti	08-10	D102	R6	pe	08-10	D102
				R7	pe	12-14	D102

Olkoon tutkittavina matriisit

$$M = \begin{pmatrix} 2 & 3 & 0 \\ 2 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{ja} \quad N = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}.$$

1. Laske a) determinantti b) transpoosi ja c) käänteismatriisi matriisille M .

$$a) \det(M) = \begin{vmatrix} 2 & 3 & 0 \\ 2 & 4 & 5 \\ 0 & 0 & 1 \end{vmatrix} = +0 - 0 + 1 \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 1 \cdot (2 \cdot 4 - 3 \cdot 2) = \underline{\underline{2}}$$

$$b) M^T = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$c) \begin{aligned} |M_{11}| &= \begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} = 4 & |M_{12}| &= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 & |M_{13}| &= \begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix} = 0 \\ |M_{21}| &= \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 & |M_{22}| &= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 & |M_{23}| &= \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} = 0 \\ |M_{31}| &= \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix} = 15 & |M_{32}| &= \begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix} = 10 & |M_{33}| &= \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2 \end{aligned}$$

$$\begin{aligned} M^{-1} &= \frac{1}{\det(M)} \begin{pmatrix} +|M_{11}| & -|M_{21}| & +|M_{31}| \\ -|M_{12}| & +|M_{22}| & -|M_{32}| \\ +|M_{13}| & -|M_{23}| & +|M_{33}| \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} +(4) & -(3) & +(15) \\ -(2) & +(2) & -(10) \\ +(0) & -(0) & +(2) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1,5 & 7,5 \\ -1 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

2. Määritä rivioperaatioiden avulla käänteismatriisi matriisille N .

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$$N = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} \textcircled{1} & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \cdot 1 \cdot 3 \\ \leftarrow \\ \leftarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 5 & -2 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} \\ \\ \downarrow \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & \textcircled{5} & -2 & -3 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array} \right) \begin{array}{l} \leftarrow + \\ \cdot \frac{1}{5} \\ :(-2) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0,6 & 0,4 & 0 & 0,2 \\ 0 & 1 & -0,4 & -0,6 & 0 & 0,2 \\ 0 & 0 & \textcircled{1} & 0,5 & -0,5 & 0 \end{array} \right) \begin{array}{l} \leftarrow - \\ \leftarrow + \\ \cdot 0,6 \quad \cdot 0,4 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0,1 & 0,3 & 0,2 \\ 0 & 1 & 0 & -0,4 & -0,2 & 0,2 \\ 0 & 0 & 1 & 0,5 & -0,5 & 0 \end{array} \right)$$

$$\rightarrow N^{-1} = \begin{pmatrix} 0,1 & 0,3 & 0,2 \\ -0,4 & -0,2 & 0,2 \\ 0,5 & -0,5 & 0 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 & 3 & 2 \\ -4 & -2 & 2 \\ 5 & -5 & 0 \end{pmatrix}$$

Tarkistus

$$N N^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0,1 & 0,3 & 0,2 \\ -0,4 & -0,2 & 0,2 \\ 0,5 & -0,5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N^{-1} N = \begin{pmatrix} 0,1 & 0,3 & 0,2 \\ -0,4 & -0,2 & 0,2 \\ 0,5 & -0,5 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Määritä adjungaatin avulla käänteismatriisi matriisille N .

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$$N = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(N) &= \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ &= (-1 - (-2)) + (1 - (-3)) + (2 - (-3)) \\ &= 1 + 4 + 5 = \underline{\underline{10}} \end{aligned}$$

$$|N_{11}| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = 1 \quad |N_{12}| = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 \quad |N_{13}| = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 5$$

$$|N_{21}| = \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -3 \quad |N_{22}| = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2 \quad |N_{23}| = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 5$$

$$|N_{31}| = \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} = 2 \quad |N_{32}| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \quad |N_{33}| = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0$$

$$N^{-1} = \frac{1}{10} \begin{pmatrix} + (1) & - (-3) & + (2) \\ - (4) & + (-2) & - (-2) \\ + (5) & - (5) & + (0) \end{pmatrix}$$

$$= \begin{pmatrix} 0,1 & 0,3 & 0,2 \\ -0,4 & -0,2 & 0,2 \\ 0,5 & -0,5 & 0 \end{pmatrix}$$

4. Ratkaise Cramerin kaavoilla yhtälöryhmä

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$$\begin{cases} x - 2y + z = -7 \\ y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 3 & 9 & -20 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 0 & -2 \\ 3 & -20 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 9 \end{vmatrix}$$

$$= (-20 - (-18)) + 2 \cdot (0 - (-6)) + (0 - 3)$$

$$= -2 + 12 - 3$$

$$= 7$$

$$D_1 = \begin{vmatrix} -7 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 9 & -20 \end{vmatrix} = +(-7) \cdot \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} - 0 + 0 = 14$$

$$D_2 = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & -2 \\ 3 & 0 & -20 \end{vmatrix} = -(-7) \cdot \begin{vmatrix} 0 & -2 \\ 3 & -20 \end{vmatrix} + 0 - 0 = 42$$

$$D_3 = \begin{vmatrix} 1 & -2 & -7 \\ 0 & 1 & 0 \\ 3 & 9 & 0 \end{vmatrix} = +(-7) \cdot \begin{vmatrix} 0 & 1 \\ 3 & 9 \end{vmatrix} - 0 + 0 = 21$$

$$x = D_1/D = 14/7 = 2$$

$$y = D_2/D = 42/7 = 6$$

$$z = D_3/D = 21/7 = 3$$

Verifioi

$$\begin{cases} x = 2 \\ y = 6 \\ z = 3 \end{cases}$$

Tarkistus:

$$\begin{cases} 2 - 2 \cdot 6 + 3 = -7 & \checkmark \\ 6 - 2 \cdot 3 = 0 & \checkmark \\ 3 \cdot 2 + 9 \cdot 6 - 20 \cdot 3 = 0 & \checkmark \end{cases}$$

5. Etsi jokin ei-triviaali ratkaisu yhtälöryhmälle

$$\begin{cases} 3x - 3y + 4z = 0 \\ y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 9 & -20 & 0 \end{array} \right) \begin{array}{l} \cdot 1 \\ \leftarrow \\ \leftarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 12 & -24 & 0 \end{array} \right) \begin{array}{l} \cdot 12 \\ \leftarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Asuetaan $z = a \rightarrow$

$$\begin{cases} 3x - 3y + 4z = 0 & (1) \\ y - 2z = 0 & (2) \\ z = a & (3) \end{cases}$$

$$(3) \rightarrow z = a$$

$$(2) \rightarrow y = 2a$$

$$(1) \rightarrow 3x - 3 \cdot 2a + 4a = 0$$

$$3x = 2a$$

$$x = \frac{2}{3}a$$

Valitaan lopulta $a = 3$

$$\rightarrow x = 2$$

$$y = 6$$

$$z = 3$$