YUKAWA POTENTIAL, PANHARMONIC MEASURE AND BROWNIAN MOTION

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Abstract. The panharmonic measure is a generalization of the harmonic measure for the solutions of the Yukawa partial differential equation. We show that the panharmonic measure shares many of the important properties of the classical harmonic measure. In particular, we show that there are natural stochastic definitions for the panharmonic measure in terms of the Brownian motion and that the harmonic and the panharmonic measures are all mutually equivalent. Furthermore, we calculate their Radon–Nikodym derivatives explicitly for some balls, yielding algorithms for numerical approximations of the solutions to the Yukawa PDE. We discuss how to simulate the Yukawa PDE with random walk on spheres and random walk on moving spheres.

1. Introduction and Preliminaries

The harmonic measure is a fundamental tool in geometric function theory, and it has interesting applications in the study of bounded analytic functions, quasiconformal mappings and potential theory. For example, the harmonic measure has proven very useful in study of quasidisks and related topics (see e.g. [1, 13, 18]). Results involving the harmonic measure have been given by numerous authors since 1930's (see [12] and references therein). In this paper we shall consider the panharmonic measure, which is a natural counterpart of the classical harmonic measure, where the harmonic functions related are replaced with the smooth solutions to the *Yukawa equation*

$$\Delta u(x) = \mu^2 u(x), \quad \mu^2 > 0. \tag{1.1}$$

The equation (1.1) first arose from the work of the Japanese physicist Hideki Yukawa in particle physics. Here $u\colon D \to \mathbb{R}$ is a two times differentiable function and $D \subset \mathbb{R}^n$, $n \geq 2$, is a domain. The Yukawa equation was first studied in order to describe the nuclear potential of a point charge. This model led to the concept of the Yukawa potential (also called a screened Coulomb potential), which satisfies an equation of the type (1.1). The Yukawa equation also arises from certain problems related to optics, see [15]. Obviously, when $\mu = 0$ we have the Laplace equation and, indeed, the results given in this paper reduce to the classical ones.

Using the terminology of Duffin [9, 10], we call a function $u\colon D \to \mathbb{R}$ *panharmonic* in a domain D if its second derivatives are continuous and

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it satisfies the Yukawa equation (1.1) for all $x \in D$. The function u is called panharmonic at $x_0 \in D$ if there is a neighborhood of x_0 where u is panharmonic.

The panharmonic, or μ-panharmonic measure, is a generalization of the harmonic measure:

1.2. **Definition.** Let $D \subset \mathbb{R}^n$ be a regular domain and let $\mu^2 \geq 0$. The *μ-panharmonic measure* on a boundary ∂D with a pole at $x \in D$ is the measure $H_\mu^x(D;\cdot)$ such that any *μ-panharmonic function* u on D that is bounded on ∂D admits the representation

$$u(x) = \int_{y \in \partial D} u(y)\, H_\mu^x(D; \mathrm{d}y).$$

The existence and uniqueness of panharmonic measure, and the notion of regularity of a domain, will be established by Theorem 2.5 later.

In Definition 1.2 above, and in all that follows, we shall always assume that $n \geq 2$, although some results are true in the dimension $n = 1$, also.

Note that if we replace the 'killing parameter' μ^2 in the Yukawa equation (1.1) with a 'creation parameter' $\lambda < 0$ we obtain another important partial differential equation, the *Helmholtz equation*. In principle, the stochastic approaches taken in this paper can be applied to the solutions of the Helmholtz equation if the domain D is small enough compared to the parameter λ. For details, we refer to Chung and Zhao [3]. If we replace μ^2 by a (positive) function, we obtain the *Schrödinger equation*. Again, the stochastic approaches taken in this paper can be applied, in principle, to the Schrödinger equation, but the results may not be mathematically very tractable. Again, we refer to Chung and Zhao [3] for details.

The rest of the paper is organized as follows: In Section 2 we show three different connections between the panharmonic measures and the Brownian motion. The first two (Theorem 2.5 and Corollary 2.11) are essentially well-known. The third one (Corollary 2.15) is new. In Section 3 we show that the panharmonic measures and the harmonic measures are all mutually equivalent (Theorem 3.2) and provide some corollaries, viz. we provide a domination principle for the Dirichlet problem related to the Yukawa equation (Corollary 3.5) and analogs of theorems of Riesz–Riesz, Makarov and Dahlberg for the panharmonic measures (Corollary 3.6). In section 4 we consider the panharmonic measures on balls and prove an analogue of the Gauss mean value theorem, or the average property, for the panharmonic functions (Theorem 4.2) and as a corollary we obtain the Liouville theorem for panharmonic functions (Corollary 4.5). Finally, in Section 5 we list some open problems and avenues for further research.

2. Yukawa Equation and Brownian motion

Let us first recall the celebrated connection between the harmonic measure and the Brownian motion first noticed by Kakutani [16] in the 1940's: Let W be a n-dimensional standard Brownian motion for some $n \geq 2$. A domain $D \subset \mathbb{R}^n$ is *regular* if the Brownian motion does not dwell on its boundary;

more precisely, D is regular if

$$\mathbb{P}^x\left[\tau_{D^c} = 0\right] = 1, \quad \text{for all } x \in \partial D,$$

where $\mathbb{P}^x$ is the probability measure under which $\mathbb{P}^x[W(0) = x] = 1$ and

$$\tau_D = \inf\{t > 0; W(t) \in D^c\}$$

is the first hitting time of the Brownian motion in the set D^c. Then the harmonic measure is the *hitting measure*:

$$H^x(D; \mathrm{d}y) = \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y, \tau_D < \infty\right]. \tag{2.1}$$

Theorem 2.5 below is a variant of the Kakutani connection (2.1). A key ingredient in the variant is the following disintegration of the harmonic measure on the time the associated Brownian motion hits the boundary ∂D:

2.2. **Lemma.** *Let $D \subset \mathbb{R}^n$ be a regular domain and $x \in D$. Then*

$$H^x(D; \mathrm{d}y) = \int_{t=0}^{\infty} h^x(D; \mathrm{d}y, t)\,\mathrm{d}t,$$

where

$$h^x(D; \mathrm{d}y, t) = \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y \mid \tau_D = t\right] \frac{\mathrm{d}\mathbb{P}^x}{\mathrm{d}t}\left[\tau_D \le t\right] \tag{2.3}$$

is the harmonic kernel.

Proof. First, we show the existence of the regular conditional distribution

$$\mathfrak{p}^x(\mathrm{d}y \,|\, t) = \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y \mid \tau_D = t\right]. \tag{2.4}$$

For this, we note that the random vector $(W(\tau_D), \tau_D)$ can be considered as a function from a space of continuous functions that are the Brownian trajectories equipped with the metric

$$d(f, g) = \sum_{T=1}^{\infty} 2^{-T} \left\| f\mathbf{1}_{[T-1,T)} - g\mathbf{1}_{[T-1,T)} \right\|_{\infty}.$$

For Brownian trajectories the metric d is almost surely finite due to the independent increments of the Brownian motion and the Borel–Cantelli lemma. Also, with the metric d, the space of Brownian paths is a Polish space. Now, by Theorem A1.2 of [19] Polish spaces are Borel spaces. Consequently, for any fixed $x \in D$, by Theorems 6.3 and 6.4 of [19], the probability kernel (2.4) exist and is measurable with respect to t. Consequently, the harmonic kernel is measurable with respect to t.

Second, we show that the distribution of the hitting time τ_D is absolutely continuous with respect to the Lebesgue measure. Let $\varepsilon > 0$ be small enough so that $B = B(x, \varepsilon) \subset D$. Then $\tau_D = \tau_B + (\tau_D - \tau_B)$. Now, the distribution of τ_B is absolutely continuous; see, e.g., the section of Bessel processes in Borodin and Salminen [2]. Also, due to the rotation symmetry of the Brownian motion, τ_B and $\tau_D - \tau_B$ are independent. Hence, by disintegration and independence, we obtain that

$$\begin{aligned}
\mathbb{P}^x[\tau_D \in \mathrm{d}t] &= \mathbb{P}^x[\tau_B + (\tau_D - \tau_B) \in \mathrm{d}t] \\
&= \int_{s=0}^{\infty} \mathbb{P}^x[t + (\tau_D - \tau_B) \in \mathrm{d}s \,|\, \tau_B = t]\, \mathbb{P}^x[\tau_B \in \mathrm{d}t] \\
&= \int_{s=0}^{\infty} \mathbb{P}^x[t + (\tau_D - \tau_B) \in \mathrm{d}s]\, \mathbb{P}^x[\tau_B \in \mathrm{d}t] \\
&= \varphi^x(t)\, \mathbb{P}^x[\tau_B \in \mathrm{d}t].
\end{aligned}$$

Thus, the distribution of τ_D is absolutely continuous, when the distribution of τ_B is absolutely continuous.

Third, we show that the formula (2.3) holds. By disintegrating and conditioning, and by using the continuity of the distribution of τ_D, we obtain that

$$\begin{aligned}
&\mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y, \tau_D < \infty\right] \\
&= \int_{t=0}^{\infty} \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y, \tau_D \in \mathrm{d}t\right] \\
&= \int_{t=0}^{\infty} \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y \mid \tau_D = t\right] \mathbb{P}^x\left[\tau_D \in \mathrm{d}t\right] \\
&= \int_{t=0}^{\infty} \mathbb{P}^x\left[W(\tau_D) \in \mathrm{d}y \mid \tau_D = t\right] \frac{\mathrm{d}\mathbb{P}^x}{\mathrm{d}t}\left[\tau_D \le t\right]\, \mathrm{d}t.
\end{aligned}$$

The claim follows now from the Kakutani connection (2.1). □

2.5. **Theorem.** *Let $D \subset \mathbb{R}^n$ be a regular domain and let $f : \partial D \to \mathbb{R}$ be bounded. Then*

$$u(x) = \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_D} f(W(\tau_D))\,;\, \tau_D < \infty\right] \tag{2.6}$$

is a solution to the Yukawa equation $\Delta u = \mu^2 u$ on D and $u = f$ on ∂D. Moreover, if $u \in C^2(\bar{D})$ then (2.6) is the only solution to the Yukawa equation.

As a consequence, the harmonic measure admits the representation

$$H^x_\mu(D; \mathrm{d}y) = \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t}\, h^x(D; \mathrm{d}y, t)\, \mathrm{d}t, \tag{2.7}$$

where $h^x(D;\cdot,\cdot)$ is the harmonic kernel defined in (2.3).

Proof. The first paragraph of Theorem 2.5 is classical; see, e.g., [3] or [11].

To see the representation (2.7), we condition on $\{\tau_D = t\}$ and use the law of total probability:

$$\begin{aligned}
u(x) &= \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_D} f(W(\tau_D))\,;\, \tau_D < \infty\right] \\
&= \int_{y \in \partial D} f(y) \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t}\, h^x(D; \mathrm{d}y, t)\, \mathrm{d}t \\
&= \int_{y \in \partial D} f(y)\, H^x_\mu(D; \mathrm{d}y).
\end{aligned}$$

□

2.8. *Remark.* Unfortunate, even for very simple D the harmonic kernel (2.3) is quite difficult to find out. The same is true for the regular conditional distribution (2.4). For smooth boundaries ∂D one can try the following approach: If ∂D is smooth, then the harmonic kernel $h^x(D; \mathrm{d}y, t)$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}y$. Indeed, define $p : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$p(t,x) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{\|x\|^2}{2t}\right). \tag{2.9}$$

Then p is the *Brownian transition kernel*:

$$p(t, x-y)\,\mathrm{d}y = \mathbb{P}^x\left[W(t) \in \mathrm{d}y\right]$$

and, due to [14, Theorem 1] the harmonic kernel can be written as

$$h^x(D; \mathrm{d}y, t) = \frac{1}{2}\frac{\partial p}{\partial \mathrm{n}_y}(D; t, x-y)\,\mathrm{d}y,$$

where n_y is the inward normal at $y \in \partial D$ and $p(D; \cdot, \cdot)$ is the transition density of a Brownian motion that is killed when it hits the boundary ∂D, which can be written as

$$p(D; t, x-y) = p(t, x-y) - \mathbb{E}^x\Big[p\big(t-\tau_D, W(\tau_D) - y\big)\,;\, \tau_D < t\Big] \tag{2.10}$$

due to [23, formula (3) on page 34].

Consequently, for C^3 boundaries the harmonic measure admits a *Poisson kernel* representation and therefore, due to the representation (2.7) the panharmonic measure also admits a Poisson kernel representation:

$$\begin{aligned}
H^x_\mu(D; \mathrm{d}y) &= \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t} h^x(D; \mathrm{d}y, t)\,\mathrm{d}t \\
&= \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t} \frac{1}{2}\frac{\partial p}{\partial \mathrm{n}_y}(D; t, x-y)\,\mathrm{d}y\,\mathrm{d}t \\
&= \left[\frac{1}{2}\int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t} \frac{\partial p}{\partial \mathrm{n}_y}(D; t, x-y)\,\mathrm{d}t\right]\mathrm{d}y.
\end{aligned}$$

Theorem 2.5 gives an interpretation of the panharmonic measure in terms of exponentially discounted Brownian motion. Let us give a second interpretation in terms of exponentially killed Brownian motion. Indeed, exponential discounting is closely related to exponential killing. The *exponentially killed Brownian motion* W_μ is

$$W_\mu(t) = W(t)\mathbf{1}_{\{Y_\mu > t\}} + \dagger \mathbf{1}_{\{Y_\mu \le t\}},$$

where $\dagger$ is a *coffin state*[1] and Y_μ is an independent exponential random variable with mean $2/\mu^2$, i.e. $\mathbb{P}\left[Y_\mu > t\right] = e^{-\frac{\mu^2}{2}t}$. Let

$$\tau^\mu_D = \inf\left\{t > 0\,;\, W_\mu(t) \in D^c\right\}.$$

Then we have the following representation of the panharmonic measure:

[1]By convention $f(\dagger) = 0$ for all functions f.

2.11. **Corollary.** *Let $D \subset \mathbb{R}^n$ be a regular domain. Then the panharmonic measure admits the representation*

$$H^x_\mu(D;\mathrm{d}y) = \mathbb{P}^x\left[W_\mu(\tau^\mu_D) \in \mathrm{d}y\,;\,\tau^\mu_D < \infty\right]. \tag{2.12}$$

Proof. Let $f:\partial D \to \mathbb{R}$ be bounded. Then, by Theorem 2.5 and the independence of W and Y_μ,

$$\begin{aligned}
&\int_{y\in\partial D} f(y)\, H^x_\mu(D;\mathrm{d}y)\\
&= \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_D} f\left(W(\tau_D)\right); \tau_D < \infty\right]\\
&= \int_{y\in\partial D} f(y) \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t}\, \mathbb{P}^x\left[W(t)\in \mathrm{d}y, \tau_D \in \mathrm{d}t\right]\\
&= \int_{y\in\partial D} f(y) \int_{t=0}^{\infty} \mathbb{P}^x\left[Y_\mu > t\right] \mathbb{P}^x\left[W(t)\in \mathrm{d}y, \tau_D \in \mathrm{d}t\right]\\
&= \int_{y\in\partial D} f(y) \int_{t=0}^{\infty} \mathbb{P}^x\left[Y_\mu > t, W(t)\in \mathrm{d}y, \tau_D \in \mathrm{d}t\right]\\
&= \int_{y\in\partial D} f(y) \int_{t=0}^{\infty} \mathbb{P}^x\left[W_\mu(t)\in \mathrm{d}y, \tau^\mu_D \in \mathrm{d}t\right]\\
&= \mathbb{E}^x\left[f\left(W_\mu(\tau^\mu_D)\right); \tau^\mu_D < \infty\right].
\end{aligned}$$

Since f was arbitrary, the claim follows. □

The two representations, Theorem 2.5 and Corollary 2.11, for the panharmonic measures are, at least in spirit, classical. Now we give a third representation for the panharmonic measure in terms of an *escaping Brownian motion*. This representation is apparently new in spirit. The representation is due to the following *Duffin correspondence* [9]: Let $D \subset \mathbb{R}^n$ be a regular domain and let $u : D \to \mathbb{R}$. Let $I \subset \mathbb{R}$ be any open interval that contains 0. Set $\bar{D} = D \times I$ and define $\bar{u} : \bar{D} \to \mathbb{R}$ by

$$\bar{u}(\bar{x}) = \bar{u}(x,\tilde{x}) = u(x)\cos(\mu\tilde{x}). \tag{2.13}$$

2.14. **Theorem.** *The function $\bar{u}$ defined by (2.13) is harmonic on $\bar{D}$ if and only if u is μ-panharmonic on D.*

Proof. Let us first show that D is regular if and only if $\bar{D}$ is regular. Let $\bar{W} = (W, \tilde{W})$ be $(n+1)$-dimensional Brownian motion. Denote

$$\begin{aligned}
\tau &= \inf\{t > 0\,;\, W(t) \in D^c\},\\
\tilde{\tau} &= \inf\{t > 0\,;\, \tilde{W}(t) \in I^c\},\\
\bar{\tau} &= \inf\{t > 0\,;\, \bar{W}(t) \in \bar{D}^c\}.
\end{aligned}$$

Note that for $\{\tilde{\tau} = \tilde{x}\}$ to happen, $\tilde{x}$ must be an endpoint of the interval I. Then, by independence of W and $\tilde{W}$,

$$\begin{aligned}
\mathbb{P}^{x,\tilde{x}}[\bar{\tau} = 0] &= \mathbb{P}^{x,\tilde{x}}[\tau = 0, \tilde{\tau} = 0]\\
&= \mathbb{P}^{x}[\tau = 0]\mathbb{P}^{\tilde{x}}[\tilde{\tau} = 0]\\
&= \mathbb{P}^{x}[\tau = 0],
\end{aligned}$$

since I is obviously regular. This shows that $\bar{D}$ is regular if and only if D is regular.

Let us then show that u satisfies the Laplace equation if and only if $\bar{u}$ satisfies the Yukawa equation. But this is straightforward calculus:

$$\begin{aligned}
\Delta_{\bar{x}}\bar{u}(\bar{x}) &= \Delta_{x,\tilde{x}}\left[u(x)\cos(\mu\tilde{x})\right] \\
&= \cos(\mu\tilde{x})\Delta_x u(x) + u(x)\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2}\cos(\mu\tilde{x}) \\
&= \cos(\mu\tilde{x})\Delta_x u(x) - \mu^2\cos(\mu\tilde{x}) \\
&= \cos(\mu\tilde{x})\left(\Delta_x u(x) - \mu^2 u(x)\right) \\
&= 0
\end{aligned}$$

if and only if $\Delta_x u(x) = \mu^2 u(x)$. □

Let $\tilde{W}$ be a 1-dimensional standard Brownian motion that is independent of W. Then $\bar{W} = (W, \tilde{W})$ is a $(n+1)$-dimensional standard Brownian motion.

Now the idea how to use the Duffin correspondence is clear. We start the Brownian particle $\bar{W}$ and count the boundary data on the side of the cylinder $\bar{D} = D \times I$, if the Brownian motion does not escape the cylinder from the bottom or from the top. In that case we count zero in the boundary. Whence the name *escaping Brownian motion.*

2.15. **Corollary.** *Let $D \subset \mathbb{R}^n$ be a regular domain. Then the panharmonic measure admits the representation*

$$\begin{aligned}
&H_\mu^x(D;\mathrm{d}y) \\
&\quad= \mathbb{E}^{x,0}\left[\cos\left(\mu\tilde{W}(\tau_D)\right); W(\tau_D)\in\mathrm{d}y, \sup_{t\le\tau_D}|\tilde{W}(t)| < \frac{\pi}{2\mu}\right] \\
&\quad= \int_{\tilde{y}=-\frac{\pi}{2\mu}}^{\frac{\pi}{2\mu}} \cos(\mu\tilde{y})\, H^{x,0}\left(D\times\left(-\frac{\pi}{2\mu},\frac{\pi}{2\mu}\right); \mathrm{d}y\otimes\mathrm{d}\tilde{y}\right). \qquad (2.16)
\end{aligned}$$

Here we have chosen $I = (-\frac{\pi}{2\mu}, \frac{\pi}{2\mu})$ in the Duffin correspondence.

Proof. The claim follows by combining the Kakutani connection (2.1) with the Duffin correspondence (2.13) by noticing that it is enough to integrate over $\partial D \times (-\pi/(2\mu), \pi/(2\mu))$ since $\cos(\mu\tilde{y}) = 0$ on the boundary $\partial(-\pi/(2\mu), \pi/(2\mu))$. □

2.17. *Remark.* Representation (2.16) is exceptionally well-suited for calculations of the panharmonic measures on upper half-spaces $\mathbb{H}_+^n = \{x \in \mathbb{R}^n; x_n > 0\}$. Indeed, Duffin [9, Theorem 5] used it to calculate the Poisson kernel representation for panharmonic measures in the dimension $n = 2$. Similar calculations can be carried out for the general case $n \geq 2$, also.

3. Equivalence of Harmonic and Panharmonic Measures

The probabilistic interpretation provided by Corollary 2.11 implies that the harmonic measure and the panharmonic ones are equivalent. Indeed, the harmonic measure counts the Brownian particles on the boundary and the panharmonic measures count the killed Brownian particles on the boundary. But the killing happens with independent exponential random variables. So, if the Brownian motion can reach the boundary with positive probability, so can the killed Brownian motion; and vice versa. Also, it does not matter, as far as the equivalence is concerned, what is the starting point of the Brownian motion, killed or not.

Theorem 3.2 below makes the heuristics above precise. As corollaries of Theorem 3.2 we obtain a domination principle for the Dirichlet problem related to the Yukawa equation (Corollary 3.5) and analogs of theorems of Riesz–Riesz, Makarov and Dahlberg for the panharmonic measures (Corollary 3.6).

The same arguments that give the existence of the regular conditional law (2.4) in the proof of Lemma 2.2 also give the existence and measurability of the following conditional Radon–Nikodym derivative

$$Z^x_\mu(D;y) = \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_D}\,\middle|\,W(\tau_D)=y\right]. \tag{3.1}$$

3.2. **Theorem.** *Let D be a regular domain. Then all the panharmonic measures $H^x_\mu(D;\cdot)$, $\mu \geq 0, x\in D$, are mutually equivalent. The Radon-Nikodym derivative of $H^x_\mu(D;\cdot)$ with respect to $H^x(D;\cdot)$ is the function $Z^x_\mu(D;\cdot)$ given by (3.1). Moreover $Z^x_\mu(D;y)$ is strictly decreasing in μ, and $0 < Z^x_\mu(D;y) \leq 1$.*

3.3. *Remark.* By Corollary 2.11 the Radon–Nikodym derivative $Z^x_\mu(D;\cdot)$ in (3.1) can be interpreted as the probability that a Brownian motion killed with intensity $\mu^2/2$, that would exit the domain D at $y\in\partial D$, survives to the boundary ∂D:

$$Z^x_\mu(D;y) = \mathbb{P}^x\left[Y_\mu > \tau_D\,|\,W(\tau_D)=y\right], \tag{3.4}$$

where Y_μ is exponentially distributed random variable with mean $2/\mu^2$ that is independent of the Brownian motion W.

Proof of Theorem 3.2. Let $x,y\in D$ and let $D_0\subset D$ be a subdomain of D such that $x\in D_0$ and $y\in\partial D_0$. Then, the Markov property of the Brownian motion and the Kakutani connection (2.1), we have

$$H^x(D;A) = \int_{y\in\partial D_0} H^y(D;A)H^x(D_0;\mathrm{d}y)$$

for all measurable $A\subset\partial D$. This shows the harmonic measures $H^x(D;\cdot)$, $x\in D$, are mutually equivalent.

To see that $Z_\mu^x(D;\cdot)$ is the Radon–Nikodym derivative, note that, by the representation (2.7) and the Kakutani connection (2.1),

$$\begin{aligned} H_\mu^x(D;\mathrm{d}y) &= \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t}\, h^x(D;\mathrm{d}y,t)\,\mathrm{d}t \\ &= \int_{t=0}^{\infty} e^{-\frac{\mu^2}{2}t}\, \mathbb{P}^x\left[W(\tau_D)\in \mathrm{d}y, \tau_D \in \mathrm{d}t\right] \\ &= \int_{y\in\partial D} \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_D}\,\middle|\, W(\tau_D)=y\right]\mathbb{P}^x\left[W(\tau_D)\in\mathrm{d}y\right] \\ &= \int_{y\in\partial D} Z_\mu^x(D;y)\,H^x(D;\mathrm{d}y). \end{aligned}$$

Finally, the fact that $0 < Z_\mu^x(D;\cdot) \le 1$ is obvious from the representation (3.1). The fact that $Z_\mu^z(D;\cdot)$ is strictly decreasing follows immediately from the representation (3.4). □

From Theorem 3.2 we obtain immediately the following *domination principle* for the Dirichlet problem related to panharmonic functions:

3.5. **Corollary.** *Let D be a regular domain and let u_μ by μ-panharmonic and u_ν be ν-panharmonic, respectively, on D with $\nu \le \mu$. Then, $u_\nu \le u_\mu$ on ∂D implies $u_\nu \le u_\mu$ on D.*

Since domains with rectifiable boundary are regular, we obtain immediately from Theorem 3.2 the following analogs of the theorems of F. Riesz and M. Riesz, Makarov and Dalhberg (see [24], [21] and [5], respectively).

3.6. **Corollary.** *Let $\mathcal{H}^s(D;\cdot)$ be the s-dimensional Hausdorff measure on ∂D.*

(i) *Let $D \subset \mathbb{R}^2$ be a simply connected planar domain bounded by a rectifiable curve. Then $H_\mu^x(D;\cdot)$ and $\mathcal{H}^1(D;\cdot)$ are equivalent for all $\mu \ge 0$ and $x \in D$.*
(ii) *Let $D \subset \mathbb{R}^2$ be a simply connected planar domain. If $E \subset \partial D$ and $\mathcal{H}^s(D;E) = 0$ for some $s < 1$, then $H_\mu^x(D;E) = 0$ for all $\mu \ge 0$ and $x \in D$. Moreover, $H_\mu^x(D;\cdot)$ and $\mathcal{H}^t(D;\cdot)$ are singular for all $\mu \ge 0$ and $x \in D$ if $t > 1$.*
(iii) *Let $D \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then $H_\mu^x(D;\cdot)$ and $\mathcal{H}^{n-1}(D;\cdot)$ are equivalent for all $\mu \ge 0$ and $x \in D$.*

4. The Average Property for Panharmonic Measures and Functions

By using the representation (2.7) one can calculate the panharmonic measures if one can calculate the corresponding harmonic kernels. Or, equivalently, one can calculate the panharmonic measures if one can calculate the corresponding harmonic measures and the Radon–Nikodym derivatives given by (3.1).

The harmonic kernels for balls are calculated in [14]. We do not, however, present the general formula here. Instead, we confine ourselves in the case where the center of the ball and the pole of the panharmonic measure coincide, and give the Gauss mean value theorem, or the average property, for panharmonic measures. As a corollary we have the Liouville theorem for the panharmonic measures.

Let $D \subset \mathbb{R}^n$ be a regular domain. For the harmonic measure the *Gauss mean value theorem* states that a function $u : D \to \mathbb{R}$ is harmonic if and only if for all balls $B_n(x,r) \subset D$ we have the *average property*

$$u(x) = \int_{y \in \partial B_n(x,r)} u(y)\, \sigma_n(r; \mathrm{d}y),$$

where

$$\sigma_n(r; \mathrm{d}y) = \frac{\Gamma(n/2)}{2\pi^{n/2}} r^{1-n}\, \mathrm{d}y$$

is the uniform probability measure on the sphere $\partial B_n(x,r)$.

For the panharmonic measures the situation is similar to the harmonic measure: the only difference is that the uniform probability measure has to be replaced by a uniform sub-probability measure that depends on the killing parameter μ and the radius of the ball r. Indeed, denote

$$\psi_n(\mu) = \frac{\mu^\nu}{2^\nu \Gamma(\nu+1) I_\nu(\mu)}, \quad \mu > 0, \tag{4.1}$$

where $\nu = (n-2)/2$ and

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

is the modified Bessel function of the first kind of order ν.

4.2. **Theorem.** *Let $D \subset \mathbb{R}^n$ be a regular domain and let $\mu > 0$. A function $u : D \to \mathbb{R}$ is μ-panharmonic if and only if it has the average property:*

$$u(x) = \psi_n(\mu r) \int_{y \in \partial B_n(x,r)} u(y)\ \sigma_n(r; \mathrm{d}y).$$

for all open balls $B_n(x,r) \subset D$. Equivalently,

$$H^x_\mu\left(B_n(x,r); \mathrm{d}y\right) = \psi_n(\mu r)\, \sigma_n(r; \mathrm{d}y).$$

4.3. *Remark.* Theorem 4.2 states that $\psi_n(\mu r)$ is the Radon–Nikodym derivative:

$$\psi_n(\mu r) = Z^x_\mu\left(B_n(x,r); y\right) = \mathbb{E}^x\left[e^{-\frac{\mu^2}{2}\tau_{B_n(x,r)}} \middle| W\left(\tau_{B_n(x,r)}\right) = y \right].$$

Proof of Theorem 4.2. Note that we may assume that $x = 0$.

Denote by τ^n_r the first hitting time of the Brownian motion W on the boundary $\partial B_n(0,r)$. I.e., τ^n_r is identical in law with the first hitting time of the Bessel process with index $\nu = (n-2)/2$ reaches the level r when it starts from zero.

From the rotation symmetry of the Brownian motion it follows that the hitting place is uniformly distributed on $\partial B_n(0,r)$ for all hitting times t.

Consequently, by Theorem 2.5 and the independence of the hitting time τ_r^n and place $W(\tau_r^n)$

$$\begin{aligned} H_\mu^0\left(B_n(0,r);\mathrm{d}y\right) &= \mathbb{E}^0\left[e^{-\frac{\mu^2}{2}\tau_r^n}; W(\tau_r^n)\in \mathrm{d}y\right] \\ &= \mathbb{E}^0\left[e^{-\frac{\mu^2}{2}\tau_r^n}\right]\mathbb{P}^0\left[W(\tau_r^n)\in \mathrm{d}y\right] \\ &= \mathbb{E}^0\left[e^{-\frac{\mu^2}{2}\tau_r^n}\right]\sigma_n(r;\mathrm{d}y). \end{aligned}$$

The hitting time distributions for the Bessel process are well-known. By, e.g., Wendel [25, Theorem 4],

$$\mathbb{E}^0\left[e^{-\frac{\mu^2}{2}\tau_r^n}\right] = \frac{(\mu r)^\nu}{2^\nu\Gamma(\nu+1)I_\nu(\mu r)}.$$

The claim follows from this. □

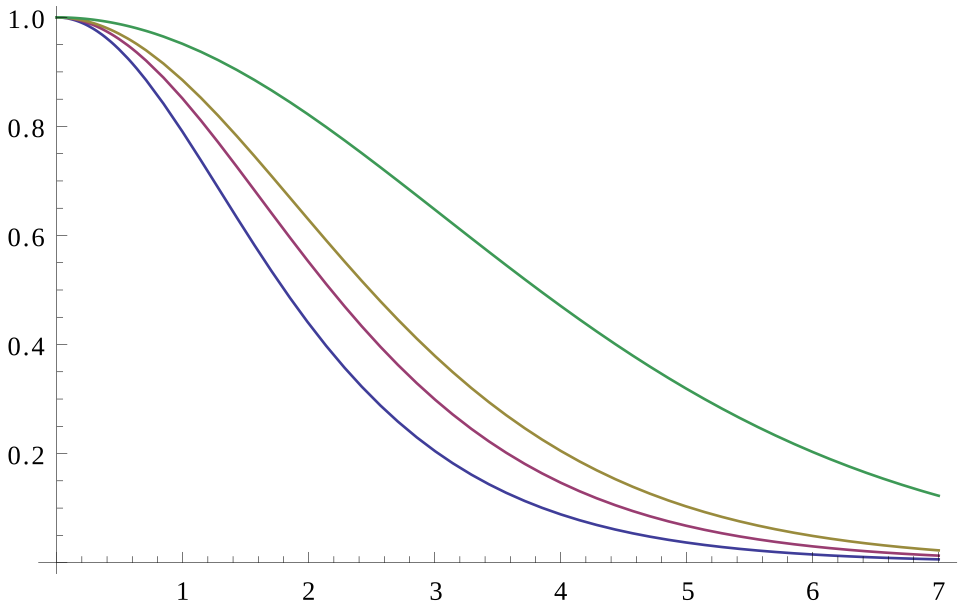
4.4. *Remark.* The Radon–Nikodym derivative, or the 'killing constant', $\psi_n(\mu)$ is rather complicated. However, some of its properties are easy to see:

(i) $\psi_n(\mu)$ is continuous in μ,
(ii) $\psi_n(\mu)$ is strictly decreasing in μ,
(iii) $\psi_n(\mu)\to 0$ as $\mu\to\infty$,
(iv) $\psi_n(\mu)\to 1$ as $\mu\to 0$,
(v) $\psi_n(\mu)$ is increasing in n.

The items (i)–(iv) are clear since $\psi_n(\mu)$ is the probability that an exponentially killed Brownian motion started from the origin with killing intensity $\mu^2/2$ is not killed before it hits the boundary of the unit ball. A non-probabilistic argument for (i)–(iv) is to note that

$$\psi_n(\mu r) = \mathbb{E}^0\left[e^{-\frac{\mu^2}{2}\tau_r^n}\right].$$

and use the monotone convergence. The item (v) is somewhat surprising: the higher the dimension n, the more likely it is for the killed Brownian motion to survive to the boundary of the unit ball. A possible intuitive explanation is that the higher the dimension the more transitive the unit ball is combined with the remarkable result by Ciesielski and Taylor [4] that probability distribution for the total time spent in a ball by $(n+2)$-dimensional Brownian motion is the same as the probability distribution of the hitting time of n-dimensional Brownian motion on the boundary of the ball.



Function ψ_n with (from bottom to top) $n = 2, 3, 4, 10$.

4.5. **Corollary.** *Let u be panharmonic on the entire space $\mathbb{R}^n$. If u is bounded, then u is constant.*

Proof. By Theorem 4.2

$$\begin{aligned}
&|u(x) - u(0)| \\
&= \left| \psi_n(\mu r) \int_{\partial B_n(x,r)} u(y)\ \sigma_n(r; \mathrm{d}y) - \psi_n(\mu r) \int_{\partial B_n(0,r)} u(y)\ \sigma_n(r; \mathrm{d}y) \right| \\
&\leq \left| \psi_n(\mu r) \int_{\partial B_n(x,r)} u(y)\ \sigma_n(r; \mathrm{d}y) \right| + \left| \psi_n(\mu r) \int_{\partial B_n(0,r)} u(y)\ \sigma_n(r; \mathrm{d}y) \right| \\
&\leq 2\psi_n(\mu r) \|u\|_\infty,
\end{aligned}$$

which tends to 0 as $r \to \infty$ by property (iii) or Remark 4.4. □

5. Discussion and Open Problems

Let us list some open problems and avenues for further studies:

I. Schrödinger equation. The Yukawa equation (1.1) is a special case of the Schrödinger equation

$$\Delta u(x) = q(x) u(x). \tag{5.1}$$

The Schrödinger equation and its connection to the Brownian motion has been studied e.g. by Chung and Zhao [3]. Our investigation here can be seen as a special case of the topic of their studies. For example, analogs of Theorem 2.5 and Corollary 2.11 are known for the Schrödinger equation. However, the Duffin correspondence (2.13) and Corollary 2.15 are not known. Moreover, the results given here cannot easily be calculated for the Schrödinger equation. The problem is that the prospective Radon–Nikodym derivate of the measure associated with the solutions of the Schrödinger equation with respect to the harmonic measure take the form

$$Z_q^x(D;y) = \mathbb{E}^x\left[e_q(\tau_D)\middle|W(\tau_D)=y\right], \tag{5.2}$$

where

$$e_q(t) = e^{-\frac{1}{2}\int_0^t q(W(s))\,\mathrm{d}s}$$

is the so-called *Feynman–Kac functional*. Thus, we see that in order to calculate the Radon–Nikodym derivative we need to know the joint density of the Feynman–Kac functional and the Brownian motion when the Brownian motion hits the boundary ∂D. If q is constant, i.e., we have either the Yukawa equation or the Helmholtz equation, then it is enough to know the joint distribution of the hitting time and place of the Brownian motion on the boundary ∂D. These distributions are well-studied, see e.g. [2, 4, 8, 14, 17, 20], but few joint distributions involving the Feynman–Kac functionals are known.

It would be interesting to calculate the Radon–Nikodym derivative (5.2) for, say, balls and half-spaces, and thus reproduce the related results of this paper to the Schrödinger equation.

II. Helmholtz equation. In addition to the Yukawa equation, the other important special case of the Schrödinger equation (5.1) is the Helmholtz equation,

$$\Delta u(x) = -\lambda u(x), \quad \lambda \geq 0. \tag{5.3}$$

It is possible to provide a Duffin correspondence for the Helmholtz equation also. Indeed, e.g., setting

$$\bar{u}(\bar{x}) = \bar{u}(x,\tilde{x}) = u(x)\cosh(\lambda\tilde{x})$$

provides a correspondence. Thus, it is reasonable to assume that our results can be extended to the Helmholtz equation (5.3) for domains that are small enough with respect to the creation parameter λ so that the associated Feynman–Kac functional is finite:

$$\mathbb{E}^x\left[e^{\frac{\lambda}{2}\tau_D}\right] < \infty. \tag{5.4}$$

It would be interesting to see if one can reproduce the results of this paper to the Helmholtz equation (5.3) for domains satisfying (5.4).

III. Panharmonic Measures on Balls. Let $B_n = B_n(0,1)$ be the n-dimensional unit ball. The harmonic measure has a nice tractable formula for (unit) balls:

$$H^x\left(B_n;\mathrm{d}y\right) = \frac{1-|x|^2}{|x-y|^n}\,\sigma_n(1;\mathrm{d}y)$$

To have a tractable formula for the panharmonic measures on balls we need a tractable formula for the exit time and place distribution

$$\mathbb{P}^x\left[W\left(\tau_{B_n}\right)\in\mathrm{d}y\,,\,\tau_{B_n}\in\mathrm{d}t\right].$$

There is a formula for this joint distribution due to Hsu [14]. The formula is rather complicated, so the calculations for the panharmonic measures $H_\mu^x(B_n;\mathrm{d}y)$ may turn out to be rather demanding and it is unclear if a tractable formula can be found.

IV. Kelvin Transformation. The Kelvin transformation is

$$\mathrm{K}_n[u](x) = |x|^{2-n} u(x^*),$$

where

$$x^* = \begin{cases} x/|x|^2, & \text{if } x \neq 0, \infty, \\ 0, & \text{if } x = \infty, \\ \infty, & \text{if } x = 0. \end{cases}$$

The Kelvin transformation preserves the harmonic functions and it can be used, e.g., to calculate harmonic measures for balls from the harmonic measures for half-spaces, and vice versa.

It would be interesting to find out a similar transformations $\mathrm{K}_{\mu,n}$ for the panharmonic functions. In principle, this should be possible by using the Radon–Nikodym derivative (3.1).

V. Simulation. Theorem 2.5, Corollary 2.11 and Corollary 2.15 give three different ways to simulate the panharmonic measures. It would be interesting to investigate their relative strengths and weaknesses in different domains (where explicit tractable formulas are difficult or impossible to obtain).

Trivial simulation is possible, but that would require simulating the trajectories with very fine time-mesh to ensure that the Brownian motion has not crossed the boundary between the simulated time-steps. To overcome this problem, Muller [22] introduced the random walk on spheres (WOS) algorithm that can be used to simulate the Laplace equation. The WOS algorithm generates spheres inside the domain and lets the Brownian motion reach the boundary of the domain in those spheres. Unfortunately, the classical WOS cannot be used to simulate the Yukawa equation in conjunction with Theorem 2.5, since it does not provide the time it takes for the Brownian motion to reach the boundary. Recently, this problem has been solved in [6, 7] where a moving walk on spheres (WOMS) algorithm was presented. This algorithm generates spheres inside the domain that let the Brownian motion reach the boundary just like the WOS algorithm, but the size of the spheres move in time. This lets the algorithm to simulate not only the exit position but also the exit time of the Brownian motion. The classical WOS can be used in conjunction with Corollary 2.15 since in this case the exit time is not needed. This provides an interesting connection between the WOS and WOMS algorithms.

Finally, let us note that in the simulations one would like to use importance sampling in order to have more Brownian paths in the target set in the boundary, and thus speeding up the convergence of the simulation. To use importance sampling in the three different simulation schemes provided by Theorem 2.5, Corollary 2.11 and Corollary 2.15, respectively, one must have a Girsanov-type theorem for the killed, discounted and escaping Brownian motion, respectively. This in turn would involve knowing the hitting time and place distribution of a Brownian motion with drift, which is studied e.g. in Yin and Wang [26].

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