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Fractional Brownian motion, random walks and binary market models

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Abstract. We prove a Donsker type approximation theorem for the fractional Brownian motion in the case $H > 1/2$. Using this approximation we construct an elementary market model that converges weakly to the fractional analogue of the Black–Scholes model. We show that there exist arbitrage opportunities in this model. One such opportunity is constructed explicitly.

Key words: Fractional Brownian motion, random walk, stock price model, binary market model

JEL Classification: C60, G10

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1 Introduction

The fractional Brownian motion is a continuous zero mean Gaussian process with stationary increments. The correlation of the increments is characterized by means of the so-called Hurst index, H. Unlike the standard Brownian motion the fractional one has long-range dependency property when $H > 1/2$. This property makes the fractional Brownian motion a plausible model in e.g. telecommunications and mathematical finance. In some empirical studies of financial time series it has been demonstrated that the log-returns have this long-range dependence, cf. Mandelbrot [7] and Shiryaev [12]. (For the use of the fractional Brownian motion in telecommunication theory we refer to Norros [8].) The fractional Brownian motion is not, however, a semimartingale when $H \neq \frac{1}{2}$. Therefore, one may suspect that a stock price model driven by it would admit arbitrage opportunities. Indeed, e.g. Dasgupta [4], Salopek [11] and Shiryaev [13] have constructed

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such opportunities by using the stochastic integration with respect to the fractional Brownian motion. Rogers [10] constructed the arbitrage by using the path properties of the fractional Brownian motion. In order to give a very simple example of the arbitrage connected to the fractional Brownian motion we consider a binary market model that appoximates the so-called fractional Black–Scholes model, i.e. a Black–Scholes model where the dynamics of the stock prices are not given by a standard Brownian motion, but a fractional one instead.

To construct the approximating binary market model we need a fractional analogue of the Donsker's theorem. This theorem, or the invariance principle, states that the standard Brownian motion can be approximated as a random walk consisting of i.i.d. random variables. In Sect. 2 we show that the fractional one can be approximated similarly by a "disturbed" random walk. We clarify how to use recent representations of the fractional Brownian motion in our approximation. Using the approximation introduced in Sect. 2 we construct a fractional binary market model in Sect. 3. We show that arbitrage opportunities exist even in this approximating "semimartingale" model. One such opportunity is constructed explicitly. The construction is based on the path properties of the "disturbed" random walk. Moreover, in contrast to Rogers [10] whose construction is based on the path information starting from minus infinity, we use information starting from time point zero.

2 Fractional Brownian motion as a limit of a random walk

2.1 Fractional Brownian motion

The fractional Brownian motion Z with index $H \in (0,1)$ is a continuous zero mean Gaussian process with stationary increments and covariance function

$$\mathbf{E}Z_tZ_s = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

If $H < \frac{1}{2}$ the increments of the process are negatively correlated. In case of $H > \frac{1}{2}$ they are positively correlated. When $H = \frac{1}{2}$ we have the standard Brownian motion W, i.e. independent increments. We assume that the self similarity index H satisfies $H > \frac{1}{2}$. In this case we have the following kernel representation of Z with respect to the standard Brownian motion

$$Z_t = \int_0^t z(t,s)\,dW_s \tag{1}$$

with a deterministic kernel

$$z(t,s) = c_H(H-\tfrac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,du,$$

where c_H is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}.$$

The integral (1) is defined in the pathwise sense in spite of the singularity of the kernel z at zero. This is possible because of the Hölder continuity of the paths of Z and z. For details see [9]. We interpret $z(t,s)$ to be zero whenever $s \geq t$.

2.2 Analogue of the Donsker's theorem

Weak convergence to the fractional Brownian motion has already been investigated by Beran [1] and Taqqu [14]. Their approximation schemes involve normal random variables. Dasgupta [4] proved an approximation using binary random variables and Mandelbrot and Van Ness's [6] representation of the fractional Brownian motion. Cutland et. al. [3] also showed a result of this kind by using nonstandard analysis. However, using the integral representation (1) we are able to provide a very simple approximation in terms of i.i.d. square integrable random variables.

Let W be the standard Brownian motion and $\xi_i^{(n)}$ i.i.d. random variables with $\mathbf{E}\xi^{(n)} = 0$ and $\mathbf{D}^2\xi^{(n)} = 1$. Denote

$$W_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^{(n)},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. By Donsker's theorem $W^{(n)}$ converges weakly to W in the Skorohod space (see e.g. [2]). Set

$$Z_t^{(n)} \;:=\; \int_0^t z^{(n)}(t,s)\,dW_s^{(n)} \;=\; \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt \rfloor}{n}, s)\,ds\, \frac{1}{\sqrt{n}} \xi_i^{(n)},$$

where z is the kernel that transforms the standard Brownian motion into a fractional one and for all t the function $z^{(n)}(t,\cdot)$ is an approximation to $z(t,\cdot)$, viz.

$$z^{(n)}(t,s) \;:=\; n \int_{s-\frac{1}{n}}^{s} z(\tfrac{\lfloor nt \rfloor}{n}, u)\,du.$$

Theorem 1 *The random walk $Z^{(n)}$ converges weakly to the fractional Brownian motion.*

Proof The proof consist of showing that the finite-dimensional distributions of $Z^{(n)}$ converge to those of Z and then showing that $Z^{(n)}$ is tight.

Let us consider the limiting finite-dimensional distributions. For arbitrary $a_1,\ldots,a_d \in \mathbb{R}$ and $t_1,\ldots,t_d \in [0,T]$ we want to show that

$$Y^{(n)} := \sum_{k=1}^{d} a_k Z_{t_k}^{(n)}$$

converges to a normal distribution with variance $\mathbf{E}(\sum_{k=1}^{d} a_k Z_{t_k})^2$. Let us calculate the limiting variance of $Y^{(n)}$. Denote $(\sigma^{(n)})^2 := \mathbf{D}^2 Y^{(n)}$. Now

$$(\sigma^{(n)})^2 = \sum_{k,l=1}^{d} a_k a_l \, n \sum_{i=1}^{\lfloor nT \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_l \rfloor}{n}, s)\, ds \tag{2}$$

By the mean value theorem (2) is equal to

$$\sum_{k,l=1}^{d} a_k a_l \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} z(\tfrac{\lfloor nt_k \rfloor}{n}, s_{i,k}^{(n)}) z(\tfrac{\lfloor nt_l \rfloor}{n}, s_{i,l}^{(n)}) \tag{3}$$

for some $s_{i,k}^{(n)}, s_{i,l}^{(n)} \in (\frac{i-1}{n}, \frac{i}{n}]$. Since the functions $z(t,\cdot)$ are continuous and decreasing in $(0,T]$ we obtain that the inner sum in Formula (3) is equal to

$$\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} z(\tfrac{\lfloor nt_k \rfloor}{n}, u_i^{(n)}) z(\tfrac{\lfloor nt_l \rfloor}{n}, u_i^{(n)}) \tag{4}$$

for some

$$u_i^{(n)} \in \left[\min(s_{i,k}^{(n)}, s_{i,l}^{(n)}), \max(s_{i,k}^{(n)}, s_{i,l}^{(n)})\right] \subseteq \left(\frac{i-1}{n}, \frac{1}{n}\right].$$

By using the fact that the kernel z is continuous with respect to both arguments and that the maps $t \mapsto \frac{\lfloor nt \rfloor}{n}$ converge uniformly to the identity map in $[0,T]$ we see that (4) is a Riemann type sum. It follows that (3), and hence $(\sigma^{(n)})^2$, converges to

$$\sum_{k,l=1}^{d} a_k a_l \int_0^T z(t_k, s) z(t_l, s)\, ds = \mathbf{E}(\sum_{k=1}^{d} a_k Z_{t_k})^2.$$

Let us now write $Y^{(n)}$ as a sum in i.

$$Y^{(n)} = \sum_{i=1}^{\lfloor nT \rfloor} \sqrt{n} \xi_i^{(n)} \sum_{k=1}^{d} a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds =: \sum_{i=1}^{\lfloor nT \rfloor} Y_i^{(n)}.$$

Lindeberg's condition is satisfied if for all $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{(\sigma^{(n)})^2} \sum_{i=1}^{\lfloor nT \rfloor} \mathbf{E}(Y_i^{(n)})^2 \mathbf{1}_{\{|Y_i^{(n)}| > \varepsilon \sigma^{(n)}\}} = 0. \tag{5}$$

We give an upper bound for the random variables $(Y_i^{(n)})^2$. By Cauchy–Schwartz inequality and the facts that $z(\cdot, s)$ is increasing and $z(t,\cdot)$ is decreasing we obtain

$$\begin{aligned}
(Y_i^{(n)})^2 &= n(\xi_i^{(n)})^2(\sum_{k=1}^d a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds)^2 \\
&\leq n(\xi_i^{(n)})^2 A(\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(T,s)\, ds)^2 \\
&\leq (\xi_i^{(n)})^2 A \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(T,s)^2\, ds \\
&\leq (\xi_i^{(n)})^2 A \int_0^{\frac{1}{n}} z(T,s)^2\, ds \\
&= (\xi_i^{(n)})^2 A \delta^{(n)}, \qquad (6)
\end{aligned}$$

where $A := (\sum_{k=1}^d a_k)^2$ and $\delta^{(n)} := \int_0^{1/n} z(T,s)^2\, ds$. We obtain

$$\left\{|Y_i^{(n)}| > \varepsilon\sigma^{(n)}\right\} \subseteq \left\{(\xi_i^{(n)})^2 A\delta^{(n)} > \varepsilon^2(\sigma^{(n)})^2\right\} =: D^{(n)}(\xi_i^{(n)}). \qquad (7)$$

Using inequality (6) and the inclusion (7) we obtain

$$\mathbf{E}(Y_i^{(n)})^2 \mathbf{1}_{\{|Y_i^{(n)}|>\varepsilon\sigma^{(n)}\}} \leq (\sigma_i^{(n)})^2 \mathbf{E}(\xi_i^{(n)})^2 \mathbf{1}_{D^{(n)}(\xi_i^{(n)})} =: (\sigma_i^{(n)})^2 \mathbf{E}\xi^2 \mathbf{1}_{D^{(n)}},$$

where $\xi := \xi_1^{(1)}$, $D^{(n)} := D^{(n)}(\xi_1^{(1)})$ and $(\sigma_i^{(n)})^2 := \mathbf{D}^2 Y_i^{(n)}$. Using this upper bound to the Lindeberg's condition (5) we obtain

$$\begin{aligned}
\frac{1}{(\sigma^{(n)})^2} \sum_{i=1}^{\lfloor nT \rfloor} \mathbf{E}(Y_i^{(n)})^2 \mathbf{1}_{\{|Y_i^{(n)}|>\varepsilon\sigma^{(n)}\}} &\leq \frac{(\sigma_1^{(n)})^2 + \cdots + (\sigma_{\lfloor nT \rfloor}^{(n)})^2}{(\sigma^{(n)})^2} \mathbf{E}\xi^2 \mathbf{1}_{D^{(n)}} \\
&= \mathbf{E}\xi^2 \mathbf{1}_{D^{(n)}}.
\end{aligned}$$

Since $z(T,\cdot)^2$ is integrable $\delta^{(n)}$, and consequently $\mathbf{E}\xi^2\mathbf{1}_{D^{(n)}}$, tends to zero. Hence (5) holds and the convergence of the finite-dimensional distributions follows.

It remains to prove the tightness. Let $s < t$ be arbitrary time points. By using Cauchy–Schwartz inequality we obtain

$$\begin{aligned}
\mathbf{E}(Z_t^{(n)} - Z_s^{(n)})^2 &= \mathbf{E}(\sum_{i=1}^{\lfloor nt \rfloor} \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt \rfloor}{n}, u) - z(\tfrac{\lfloor ns \rfloor}{n}, u)\, du\, \xi_i^{(n)})^2 \\
&= \sum_{i=1}^{\lfloor nt \rfloor} (\sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt \rfloor}{n}, u) - z(\tfrac{\lfloor ns \rfloor}{n}, u)\, du)^2 \\
&\leq \sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(\tfrac{\lfloor nt \rfloor}{n}, u) - z(\tfrac{\lfloor ns \rfloor}{n}, u))^2\, du \\
&\leq \int_0^t (z(\tfrac{\lfloor nt \rfloor}{n}, u) - z(\tfrac{\lfloor ns \rfloor}{n}, u))^2\, du \\
&= \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \qquad (8)
\end{aligned}$$

Let now $s < t < u$ be arbitrary. Using Cauchy–Schwartz inequality again and the bound (8) we obtain

$$\begin{aligned}
\mathbf{E}|Z_t^{(n)} - Z_s^{(n)}||Z_u^{(n)} - Z_t^{(n)}| &\leq \left(\mathbf{E}(Z_t^{(n)} - Z_s^{(n)})^2\right)^{\frac{1}{2}} \left(\mathbf{E}(Z_u^{(n)} - Z_t^{(n)})^2\right)^{\frac{1}{2}} \\
&\leq \left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^H \left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor nt\rfloor}{n}\right|^H \\
&\leq \left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}
\end{aligned}$$

If now $u - s \geq \frac{1}{n}$ we have

$$\mathbf{E}|Z_t^{(n)} - Z_s^{(n)}||Z_u^{(n)} - Z_t^{(n)}| \leq |2(u-s)|^{2H}. \tag{9}$$

If on the other hand $u - s < \frac{1}{n}$ then either s and t or t and u lie in the same subinterval $[\frac{m}{n}, \frac{m+1}{n})$ for some m. Thus the left hand side of (9) is zero. Therefore (9) holds for all $s < t < u$. Recalling now that $H > \frac{1}{2}$ and by Theorem 15.6 of Billingsley [2] we have the tightness of $Z^{(n)}$. □

Note that the increments of the random walk $Z^{(n)}$ are not independent. Also, note that the approximating kernel $z^{(n)}$ can be changed to

$$\tilde{z}^{(n)}(t,s) := \sum_{i=1}^{\lfloor nt\rfloor} z(\tfrac{\lfloor nt\rfloor}{n}, s_i^{(n)})\mathbf{1}_{(\frac{i-1}{n},\frac{i}{n}]}(s),$$

where $s_i^{(n)}$'s are real numbers belonging to the intervals $(\frac{i-1}{n}, \frac{i}{n}]$, respectively.

Denote by ΔX and $[X]$ the jump and quadratic variation process of a random walk X, respectively, i.e.

$$\Delta X_t := X_t - \lim_{s\uparrow t} X_s \quad \text{and} \quad [X]_t := \sum_{s\leq t}(\Delta X_s)^2.$$

Theorem 2 *The process* $[Z^{(n)}]$ *converges to zero in* $L^1(\mathbf{P} \times \text{Leb})$, *where* Leb *is the Lebesgue measure on the interval* $[0, T]$.

Proof Using the bound (8) from the proof of the Theorem 1 we obtain

$$\mathbf{E}(\Delta Z_t^{(n)})^2 \leq \mathbf{E}(Z_t^{(n)} - Z_{t-\frac{1}{n}}^{(n)})^2 \leq n^{-2H}.$$

Hence

$$\mathbf{E}|Z^{(n)}]_t = \sum_{s\leq t} \mathbf{E}(\Delta Z_t^{(n)})^2 \leq nt \cdot n^{-2H} = tn^{1-2H}.$$

Since $[Z^{(n)}]$ is a non-negative and increasing process we obtain

$$\int_0^T \mathbf{E}[Z^{(n)}]_t\, dt \leq \int_0^T \mathbf{E}[Z^{(n)}]_T\, dt \leq T^2 n^{1-2H}.$$

Recall that $H > \frac{1}{2}$ and the claim follows. □

Corollary 1 *The processes* $\Delta Z^{(n)}$ *and* $[Z^{(n)}]$ *converge to zero in probability.*

Consider the process $S^{(n)}$ defined by

$$S_t^{(n)} := \prod_{s\le t}(1 + \Delta Z_s^{(n)}).$$

Our aim is to prove that the process $S^{(n)}$ converges weakly to the geometric fractional Brownian motion. Since the big jumps of the process $Z^{(n)}$ are somewhat of a nuisance we shall consider them separately. Namely, define the processes $Z^{(1,n)}$ and $Z^{(2,n)}$ as follows

$$Z_t^{(1,n)} := \sum_{s\le t} \Delta Z^{(n)} \mathbf{1}_{\{|\Delta Z_s^{(n)}|<\frac{1}{2}\}} \tag{10}$$

$$Z_t^{(2,n)} := \sum_{s\le t} \Delta Z^{(n)} \mathbf{1}_{\{|\Delta Z_s^{(n)}|\ge\frac{1}{2}\}}. \tag{11}$$

In view of Corollary 1 and Theorem 4.1 of Billingsley [2] the following lemma is obvious.

Lemma 1 *The process* $Z^{(2,n)}$ *converges to zero in probability and hence* $Z^{(1,n)}$ *converges weakly to* Z.

Theorem 3 *The process* $S^{(n)}$ *converges weakly to the geometric fractional Brownian motion* e^Z.

Proof Write

$$S^{(n)} = S^{(1,n)}S^{(2,n)},$$

where

$$S_t^{(i,n)} := \prod_{s\le t}(1 + \Delta Z^{(i,n)})$$

for $i = 1, 2$ and the processes $Z^{(i,n)}$ are as in (10) – (11).

By Theorem 4.1 and Problem 1 p. 28 of Billingsley [2] it is enough to show that $S^{(1,n)}$ converges weakly to e^Z and $S^{(2,n)}$ converges to 1 in probability.

Let us first consider the process $S^{(2,n)}$. Let $\varepsilon > 0$. Then

$$\begin{aligned}
\mathbf{P}(\sup_{t\le T} |S_t^{(2,n)} - 1| \ge \varepsilon) &= \mathbf{P}(\sup_{t\le T} |\prod_{s\le t}(1 + \Delta Z_s^{(2,n)}) - 1| \ge \varepsilon) \\
&\le \mathbf{P}(\sup_{t\le T} |\Delta Z_t^{(2,n)}| > 0) \\
&= \mathbf{P}(\sup_{t\le T} |\Delta Z_t^{(n)}| > \tfrac{1}{2}).
\end{aligned}$$

Since the process $\Delta Z^{(n)}$ converges to zero in probability by Corollary 1 the convergence of $S^{(2,n)}$ to one in probability follows.

It remains to prove that $S^{(1,n)}$ converges weakly to e^Z. Let us first prove that $\log S^{(1,n)}$ converges weakly to Z. Since $|\Delta Z_t^{(1,n)}| < \frac{1}{2}$ for all $t \le T$ the logarithm of $S^{(1,n)}$ is well defined. According to Taylor's theorem

$$\log(1+x) = x - \frac{1}{2}x^2 + r(x)x^2,$$

where $r(x)$ tends to zero as x tends to zero. Hence

$$\begin{aligned}\log S_t^{(1,n)} &= \sum_{s\le t}\left(\Delta Z_s^{(1,n)} - \frac{1}{2}(\Delta Z_s^{(1,n)})^2 + r(\Delta Z_s^{(1,n)})(\Delta Z_s^{(1,n)})^2\right)\\ &= Z_t^{(1,n)} - \frac{1}{2}[Z^{(1,n)}]_t + \sum_{s\le t} r(\Delta Z_s^{(1,n)})(\Delta Z_s^{(1,n)})^2.\end{aligned}$$

Now $[Z^{(n)}]$ converges to zero in probability by Corollary 1. Obviously $[Z^{(1,n)}] \le [Z^{(n)}]$. So, $[Z^{(1,n)}]$ also converges to zero in probability. Since $\Delta Z_t^{(1,n)} < \frac{1}{2}$ the remainder r is uniformly bounded. Hence the third term converges also to zero in probability by using the Corollary 1 again. Hence we obtain, by using the Theorem 4.1 of Billingley [2] and Lemma 1, that $\log S^{(1,n)}$ converges weakly to Z.

Finally, since the exponential is a continuous functional (in the Skorokhod topology) the theorem follows. □

3 Fractional Brownian motion and binary market models

3.1 Fractional Black–Scholes model

Consider two assets, or securities, that are traded continuously over the time interval $[0,T]$. Here 0 is the current date and the terminal date T is fixed.

Denote by B the riskless asset, or *bond.* The dynamics of the asset B are

$$dB_t = r_t dt B_t, \tag{1}$$

where r is a deterministic interest rate.

The risky asset, *stock,* is denoted by S and has the dynamics

$$dS_t = (a_t dt + \sigma dZ_t)S_t, \tag{2}$$

where $\sigma > 0$ is a constant and Z is a fractional Brownian motion with index $H > \frac{1}{2}$. The function a is the deterministic drift of the stock.

Assume that r and a are continuously differentiable on the interval $[0,T]$ then the solutions of the (stochastic) differential Eqs. (1) – (2) are given by

$$B_t = B_0 \exp(\int_0^t r_s\, ds) \quad \text{and} \quad S_t = S_0 \exp(\int_0^t a_s\, ds + \sigma Z_t),$$

respectively. For details we refer to Zähle [15].

3.2 Binary market models in general

Let us briefly define what we mean by binary market models. For a detailed treatment of this subject we refer to [5] and Sect. II.1e of Shiryaev [12].

Consider a securities market in which the two assets are traded at successive time periods $0 = t_0 < t_1 \cdots < t_N = T$. The dynamics of the bond and stock are now given as

$$B_n = (1 + r_n)B_{n-1} \tag{3}$$

and

$$S_n = (a_n + (1 + X_n))\, S_{n-1}, \tag{4}$$

respectively. Here B_n and S_n are is the values of the bond and stock, respectively, over the time interval $[t_n, t_{n+1})$. Similarly r_n and a_n are the interest rate and the drift of the stock, respectively, in the corresponding interval. The sequence $X = (X_n)$ is a stochastic process such that at each time point n the random variable X_n has two possible values u_n and d_n where $d_n < u_n$. Note that the values of u_n and d_n may depend on the path of X up to time $n-1$. So the stock price S_n occupies one of the 2^n states at time n. Note that all the states are not necessarily different. However, there are 2^n different possible paths for the stock price to evolve up to time n.

By Proposition 3.6.2 of Dzhaparidze [5] a binary market excludes arbitrage opportunities if and only if for all $n = 1, \ldots, N$ we have

$$d_n < r_n - a_n < u_n. \tag{5}$$

This is connected to the existence of the so-called “equivalent martingale measure” in the following way. Let $\mathbf{P}$ be the law of X in (4). We want to find a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that $\frac{S}{B}$ is a $\mathbf{Q}$-martingale. It is easy to see that such $\mathbf{Q}$ must satisfy

$$\mathbf{Q}(X_n = u_n | X_1, \ldots, X_{n-1}) = \frac{r_n - a_n - d_n}{u_n - d_n} \in (0, 1). \tag{6}$$

Obviously, the conditions (5) and (6) are the same. Moreover, condition (6) defines a unique martingale measure, i.e. the binary market models are complete. For details, see Chap. 3 of Dzhaparidze [5]

3.3 Fractional binary market model

We define a binary model that approximates the fractional Black–Scholes model described in Sect. 3.1. In particular, we construct sequences $B^{(N)}$ and $S^{(N)}$ corresponding to the bond and stock dynamics given in Formulas (3) – (4) that converge weakly to the dynamics given in Formulas (1) – (2) as N tends to infinity.

We introduce some notations. Denote

$$k(n,i) := k^{(N)}(n,i) := \sqrt{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N}, s)\, ds,$$

where z is the kernel introduced in Sect. 2. Define the stochastic process X in Formula (4) by

$$X_n := X_n^{(N)} := \sigma \Delta Z_{Tn/N}^{(N)},$$

where $Z^{(N)}$ is the approximation of fractional Brownian motion defined in Sect. 2. The constant $\sigma > 0$ is the volatility of the stock in the Formula (2). Let us write the process X by using the kernel k.

$$X_n = \sigma \sum_{i=1}^{n} (k(n,i) - k(n-1,i))\, \xi_i, \tag{7}$$

where the random variables $\xi_i = \xi_i^{(N)}$ are i.i.d. and binary, i.e. for all i

$$\mathbf{P}(\xi_i = 1) = \frac{1}{2} = \mathbf{P}(\xi_i = -1).$$

Setting

$$f_{n-1}(x_1, \ldots, x_{n-1}) = \sigma \sum_{i=1}^{n-1} (k(n,i) - k(n-1,i))\, x_i$$

to denote the contribution of the $n-1$ first jumps of the random walk and

$$g_n(x) = \sigma k(n,n) x$$

to denote the contribution of the last jump we can write

$$X_n = f_{n-1}(\xi_1, \ldots, \xi_{n-1}) + g_n(\xi_n),$$

when $n \geq 1$ using the conversion $f_0 = 0$. Since the ξ_i's are a binary we obtain

$$\begin{aligned} u_n &= u_n(\xi_1, \ldots, \xi_{n-1}) = f_{n-1}(\xi_1, \ldots, \xi_{n-1}) + g_n(1) \\ d_n &= d_n(\xi_1, \ldots, \xi_{n-1}) = f_{n-1}(\xi_1, \ldots, \xi_{n-1}) + g_n(-1). \end{aligned}$$

Define the deterministic sequences $r^{(N)}$ and $a^{(N)}$ in the Formulas (3) – (4) by

$$r_n^{(N)} := \frac{1}{N} r_{Tn/N} \quad \text{and} \quad a_n^{(N)} := \frac{1}{N} a_{Tn/N},$$

respectively, where r and a are as in the fractional Black–Scholes model.

In view of Theorem 3 it is now clear that this model approximates the fractional Black–Scholes model in the following sense.

Theorem 4 *The price processes $B^{(N)}$ and $S^{(N)}$ converge weakly to the corresponding price processes B and S in the fractional Black–Scholes model.*

Let us now show that even this approximative model is not free of arbitrage.

Theorem 5 *The fractional binary market admits arbitrage opportunities.*

Proof It is enough to show that condition (5) does not hold, i.e. the condition

$$f_{n-1}(\xi_1,\ldots,\xi_{n-1}) + g_n(-1) < r_n - a_n < f_{n-1}(\xi_1,\ldots,\xi_{n-1}) + g_n(1)$$

fails for some sequence $(\xi_1,\ldots,\xi_n)$. In particular, by symmetry it is enough to show that

$$f_{n-1}(1,\ldots,1) - g_n(1) \;\geq\; 0$$

for some $n \geq 2$ or

$$\sum_{i=1}^{n-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s) - z(\tfrac{n-1}{N},s)\,ds - \int_{\frac{n-1}{N}}^{\frac{n}{N}} z(\tfrac{n}{N},s)\,ds \;\geq\; 0. \tag{8}$$

Let us first give a lower bound for the integral term under the sum in inequality (8). Denote $C := c_H(H-\frac{1}{2})$ and $\alpha := H - \frac{1}{2}$.

$$\begin{aligned}
&\int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s) - z(\tfrac{n-1}{N},s)\,ds \\
&\quad= C \int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha} \int_{\frac{n-1}{N}}^{\frac{n}{N}} u^{\alpha}(u-s)^{\alpha-1}\,du\,ds \\
&\quad\geq \frac{C}{\alpha}\left(\frac{n-1}{N}\right)^{\alpha} \int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha}\left\{\left(\frac{n}{N}-s\right)^{\alpha} - \left(\frac{n-1}{N}-s\right)^{\alpha}\right\} ds \\
&\quad\geq \frac{C}{\alpha}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n-i}{N}\right)^{\alpha} - \left(\frac{n-1-i}{N}\right)^{\alpha}\right\} \int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha}\,ds \\
&\quad\geq \frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n+1-i}{i}\right)^{\alpha} - \left(\frac{n-i}{i}\right)^{\alpha}\right\}.
\end{aligned} \tag{9}$$

By using the lower bound (9) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s) - z(\tfrac{n-1}{N},s)\,ds \\
&\quad\geq \sum_{i=1}^{n-1} \frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n+1-i}{i}\right)^{\alpha} - \left(\frac{n-i}{i}\right)^{\alpha}\right\} \\
&\quad\geq \frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha} (n-1)^{-\alpha} \sum_{i=1}^{n-1} \{(n+1-i)^{\alpha} - (n-i)^{\alpha}\} \\
&\quad= \frac{C}{\alpha} N^{-\alpha-1}(n^{\alpha}-1).
\end{aligned} \tag{10}$$

We give an upper bound for the latter term in the right hand side of (8).

$$
\begin{aligned}
\int_{\frac{n-1}{N}}^{\frac{n}{N}} z(\tfrac{n}{N},s)\,ds &= C\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\int_s^{\frac{n}{N}} u^{\alpha}(u-s)^{\alpha-1}\,du\,ds \\
&\le C\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\left(\frac{n}{N}\right)^{\alpha}\int_s^{\frac{n}{N}} (u-s)^{\alpha-1}\,du\,ds \\
&= \frac{C}{\alpha}\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\left(\frac{n}{N}\right)^{\alpha}\left(\frac{n}{N}-s\right)^{\alpha} ds \\
&\le \frac{C}{\alpha(\alpha+1)}\left(\frac{n}{n-1}\right)^{\alpha} N^{-\alpha-1}. \qquad (11)
\end{aligned}
$$

Since (10) tends to infinity and (11) tends to $\frac{C}{\alpha(\alpha+1)}N^{-\alpha-1}$ we obtain (8) for all $n \ge n_H$ and the theorem follows. □

Note that n_H grows to infinity as H tends to $\frac{1}{2}$. In particular, when $H = 0.6$ we have $n_H \approx 350$. This estimate was obtained by using the bounds (9) and (11).

Let us construct one arbitrage opportunity explicitly. Suppose first that the discounted drift $\tilde{a} := a - r$ satisfies $\tilde{a}_{n_0} < 0$ for some $n_0 \ge n_H$. In this case do as follows: if the stock price only takes jumps down up to time $n_0 - 1$, sell short M stocks and put the money into the bond. Since now $u_{n_0} < 0$ by (8) we have

$$MS_{n_0+1} < MS_{n_0}(1+\tilde{a}_n) < MS_{n_0}.$$

Hence your wealth at time $n_0 + 1$ is positive. If on the other hand $\tilde{a}$ remains non-negative, construct the strategy as follows: if the stock price has taken only upward jumps up to time $n \ge n_H - 1$, buy M stocks. Now $d_n > 0$ by (8). We obtain

$$MS_{n+1} > MS_n.$$

Your wealth is positive a time $n + 1$.

Finally, note that up to time n_H there are at least two paths admitting arbitrage opportunities, viz. $(-1,\dots,-1)$ and $(1,\dots,1)$. Since these paths do not depend on N, provided $N \ge n_H$, we obtain that

$$\lim_{N\to\infty} \frac{\#\text{ abritrage paths}}{2^N} \ \ge\ \frac{2}{2^{n_H-1}} \ =\ 2^{2-n_H} \ >\ 0.$$

4 Conclusions

Our binary approximation of the fractional Black–Scholes model admits arbitrage opportunities. This is due to the long-range dependence of the fractional Brownian motion with $H > \frac{1}{2}$. Heuristically, if the stock price had an upward trend long enough it will keep increasing for some time. Note that using (5) one can always determine from the data whether there is an arbitrage opportunity at hand or not.

In order to get an arbitrage free fractional Black–Scholes model we must introduce e.g. transaction costs or costly information. One can also introduce a suitable predictable interest rate. For some discussion we refer to [11].

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