# FRACTIONAL BROWNIAN MOTION AS A MODEL IN FINANCE

T. SOTTINEN AND E. VALKEILA

Abstract. In the classical Black & Scholes pricing model the randomness of the stock price is due to Brownian motion $W$. It has been suggested that one should replace the standard Brownian motion by a fractional Brownian motion $Z$. It is known that this will introduce some problems, e.g. related to arbitrage. We give a survey of some recent work in connection to this problem. We end by giving a suggestion how to price European options in this fractional pricing model.

**JEL Classification:** C60, G10

**Mathematics Subject Classification (1991):** 60F17, 60G15, 90A09

## 1. Classical Black & Scholes pricing model

Recall the classical Black & Scholes pricing model with two assets, the riskless bond and the risky stock. The randomness of the stock price $S$ is due to Brownian motion $W$ and the bond price $B$ is deterministic with a constant interest rate. The dynamics of the prices are

$$\begin{aligned} \mathrm{d}S_t &= S_t\left(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right), \\ \mathrm{d}B_t &= B_t r\,\mathrm{d}t \end{aligned}$$

with $B_0 = 1$ and $S_0$ is a positive constant. The parameters $\mu \in \mathbb{R}$, $r, \sigma > 0$ are supposed to be known. Traditionally one assumes that there are no dividends, no transaction costs, same interest rate $r$ for lending and saving on the bond and no limitations on short-selling of the stock.

Note that in what follows one can easily replace the constants $\mu$, $r$, and $\sigma$ with deterministic functions. We consider the constant case only for notational convenience. If, however, one allows stochasticity in the parameters $r$ or $\sigma$ the situation changes dramatically.

1.1. **Completeness.** As well-known the Black & Scholes model is complete (and free of arbitrage in the class of so-called admissible strategies to be considered later). This is due to the fact that there exists a unique risk neutral measure $\mathbf{Q}$, equivalent to the real world measure $\mathbf{P}$, such that the discounted stock price process $S/B$ is a martingale under this measure. The measure $\mathbf{Q}$ is identified by the Girsanov theorem (cf. Shiryaev [24, p. 673]) as

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\Big|\mathscr{F}_t^W = \exp\left(-\frac{\mu - r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t\right).$$

*Date*: October 13, 2001.

Here $\mathscr{F}_t^W$ is the $\sigma$-algebra generated by the Brownian motion $W$, or equivalently by the stock price $S$, upto time $t \leq T$. The constant $\frac{\mu - r}{\sigma}$ is often referred to as the *market price of risk.*

The fair price $C_T(f)$ of a claim $f$ on the stock price $S$ expiring at time $T$ is given by

$$C_T(f) = \mathbf{E}_{\mathbf{Q}} \frac{f}{B_T}. \tag{1.1}$$

The fairness of the price (1.1) follows from a hedging argument. Let $\pi = (\pi_t)_{t\in[0,T]}$ be a portfolio process, i.e. $\pi_t$ denotes the number of the shares of the stock owned by an investor a time $t$. Assume that $\pi$ is self-financing which is to say that the value process $V = V^\pi$ of the portfolio $\pi$ satisfies

$$\mathrm{d}V_t = \pi_t \,\mathrm{d}S_t + (V_t - \pi_t S_t)\,\mathrm{d}B_t. \tag{1.2}$$

It follows that the discounted value process $V/B$ is a local $\mathbf{Q}$-martingale. Assume that it is a (proper) $\mathbf{Q}$-martingale, i.e. the corresponding portfolio $\pi$ is *admissible.* Thus, if $V$ replicates the claim $f$, we must have

$$\frac{V_t}{B_t} = \mathbf{E}_{\mathbf{Q}}\left[\frac{f}{B_T}\middle|\mathscr{F}_t^W\right].$$

This shows that (1.1) is $V_0$, the capital needed to hedge, or replicate, the claim $f$. Moreover, the hedging portfolio $\pi$ can be constructed by using the Ito–Clark–Ocone formula. Indeed, the martingale representation theorem (cf. Shiryaev [24, p. 257]) tells us that

$$\frac{V_t}{B_t} = V_0 + \int_0^t \gamma_s \,\mathrm{d}\tilde{W}_s,$$

where $\gamma$ is a predictable stochastic process and $\tilde{W}$ is a Brownian motion under the measure $\mathbf{Q}$, i.e. (by the Girsanov theorem)

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t.$$

Plugging in the self-financing condition (1.2) we obtain

$$\frac{V_t}{B_t} = V_0 + \sigma \int_0^t \pi_s \frac{S_s}{B_s} \,\mathrm{d}\tilde{W}_s.$$

Thus,

$$\pi_t = \gamma_t \frac{B_t}{\sigma S_t}.$$

The Ito–Clark–Ocone formula (cf. Karatzas and Shreve [10, p. 369]) gives us the process $\gamma$ in terms of the so-called Malliavin derivative $D_t$ as

$$\gamma_t = \mathbf{E}_{\mathbf{Q}}\left[D_t \frac{f}{B_T}\middle|\mathscr{F}_t^W\right].$$

Putting everything together we have a formula for the hedging portfolio

$$\pi_t = \frac{B_t}{B_T \sigma S_t} \mathbf{E}_{\mathbf{Q}}\big[D_t f | \mathscr{F}_t^W\big]\,. \tag{1.3}$$

1.2. **Arbitrage.** Let us now consider the arbitrage in the Black & Scholes pricing model. If the portfolio, or strategy, $\pi$ is admissible then the corresponding value process is a martingale. Hence, if we start with zero capital and assume that $V_T \geq 0$ then

$$0 = V_0 = \mathbf{E}_{\mathbf{Q}} \frac{V_T}{B_T} = \mathbf{E}_{\mathbf{Q}} \frac{V_T}{B_T} \mathbf{1}_{\{V_T > 0\}}.$$

So, we must have $\mathbf{Q}(V_T > 0) = 0$. Since $\mathbf{Q}$ is equivalent to $\mathbf{P}$ it follows that $\mathbf{P}(V_T > 0) = 0$. Therefore, there is no free lunch with admissible strategies.

One should note that sometimes the word admissible is used in the sense that $V$ is non-negative. Also, if the process $V/B$ is bounded from below by a non-random constant, depending possibly on $\pi$, then the portfolio $\pi$ is called *tame.* In any of these cases there are no arbitrage opportunities.

One can construct arbitrage opportunities with the so-called *doubling strategy,* however. The following example is from Karatzas and Shreve [10, p. 9]. Let $r = \mu = 0$ and $\sigma = 1$. Consider the stochastic integral

$$I_t := \int_0^t \sqrt{\frac{1}{T-s}}\, \mathrm{d}W_s.$$

The process $I$ is a martingale with the bracket

$$\langle I \rangle_t = \log\left(\frac{T}{T-t}\right).$$

The time-changed stochastic integral

$$\tilde{I}_t := I_{T - Te^{-t}}$$

has the bracket $\langle \tilde{I} \rangle_t = t$. Thus, it is a Brownian motion. Consequently,

$$\limsup_{t \to T} I_t = \limsup_{t \to \infty} \tilde{I}_t = \infty.$$

Therefore, for any $\alpha > 0$

$$\tau_\alpha := \inf\{t \in [0,T) : I_t = \alpha\} \wedge T$$

satisfies $\tau_\alpha \in (0,T)$ almost surely. Let us define a self-financing portfolio $\pi$ as

$$\pi_t := \frac{1}{S_t \sqrt{T-t}} \mathbf{1}_{(0,\tau_\alpha]}(t). \tag{1.4}$$

Then the value process $V$ satisfies

$$V_t = \int_0^t \pi_s S_s \,\mathrm{d}W_s = I_{t \wedge \tau_\alpha}.$$

So, $V_T = \alpha$ almost surely. Since $V_0 = 0$ we have constructed arbitrage. It should be noted that the strategy (1.4) is not bounded from below.

1.3. **Problems.** The Black & Scholes pricing model is very satisfactory from the theoretical point of view. Claims can be priced fairly and (in principle) one can even calculate the corresponding hedging portfolios by using the formula (1.3). Also, there are no arbitrage opportunities is the class of admissible portfolios. However, there is a problem with this model. It stipulates that the log-returns

$$\begin{aligned} R_{t_k} &:= \log \frac{S_{t_k}}{S_{t_{k-1}}} \\ &= (\mu - \frac{\sigma^2}{2})(t_k - t_{k-1}) + \sigma(W_{t_k} - W_{t_{k-1}}) \end{aligned}$$

are independent normal random variables.

The dependence structure of the log-returns have been studied using the so-called Hurst parameter $H$. In the uncorrelated case one should have $H = \frac{1}{2}$. However, many studies have indicated Hurst indices $\hat{H} > \frac{1}{2}$. E.g. for the daily exchange rate between USD and JPY between January 1972 and December 1990 the estimated Hurst index is $\hat{H} = 0.642$. For references to these studies see e.g. Peters [19] and Shiryaev [24].

There are also empirical studies indicating that the *log-returns are not normal.* This is more evident, if the observation intervals $t_k - t_{k-1}$ are short.

To overcome with the first critical point, the independence assumption of the log-returns, it has been proposed that one should replace the Brownian motion by a fractional Brownian motion which captures the long-range dependency property measured by $H$. The first one to suggest this was Mandelbrot, already in late sixties (cf. Mandelbrot [14]).

The second critical point, the non-normality of log-returns, will be completely ignored in what follows.

## 2. Long-range dependence and self-similarity

In this section we consider briefly the concepts of statistical long-range dependence and self-similarity. For a detailed discussion we refer to Beran [1].

2.1. **Long-range dependence.** Consider a stationary sequence $X = (X_k)_{k\in\mathbb{N}}$. Such a sequence is said to exhibit the statistical *long-range dependency property* if its autocorrelation function $\rho$ satisfies

$$\lim_{k\to\infty} \frac{\rho(k)}{c_\rho k^{-\alpha}} = 1 \tag{2.1}$$

for some constants $c_\rho$ and $\alpha \in (0,1)$. This is to say that the dependence between $X_k$ and $X_{k+n}$ decays slowly as $n$ tends to infinity. In particular,

$$\sum_{k=0}^{\infty} \rho(k) = \infty. \tag{2.2}$$

Actually, sometimes one defines the long-range dependence by the property (2.2) instead of (2.1).

There is also a definition involving the spectral density. Suppose that $f$ is the spectral density of $X$. If there exists constants $c_f$ and $\beta \in (0,1)$ such that

$$\lim_{\lambda \to 0} \frac{f(\lambda)}{c_f |\lambda|^{-\beta}} = 1, \tag{2.3}$$

then $X$ exhibits the statistical long-range dependency property.

The definitions (2.1) and (2.3) are connected in the following way: let $H \in (\frac{1}{2}, 1)$. Then $\alpha = 2 - 2H$ and $\beta = 2H - 1$ (cf. Beran [1, p. 43]).

2.2. **Self-similarity.** Self-similar processes were introduced by Kolmogorov [12] as models for turbulence in the early forties. In the late sixties Mandelbrot and Van Ness [15] made the concept known among the statisticians.

A centered stochastic process process $X = (X_t)_{t \in [0,T]}$ is said to be statistically *self-similar* with Hurst exponent $H$ if

$$(X_t)_{t \in [0,T]} \stackrel{d}{=} (a^{-H} X_{at})_{t \in [0,T]}$$

for all $a > 0$. Here $d$ denotes the equivalence in distribution.

If, in addition, the process $X$ is square integrable with stationary increments it follows that

$$\mathbf{Cov}(X_t, X_s) = \frac{\mathbf{Var}\, X_1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right). \tag{2.4}$$

Note that if $H = 0$ then one must have $X = 0$ identically. The case $H = 1$ is hardly interesting, since it implies $\mathbf{Corr}(X_t, X_1) = 1$. The cases $H < 0$ and $H > 1$ are impossible. It the former case $\mathbf{Var}\, X_0 = \infty$. In the latter case one would have $\mathbf{Corr}(X_t, X_1) > 1$ given $t$ big enough. Hence, one assumes that $H \in (0,1)$ in the equation (2.4) above.

The connection to long-range dependence is then the following. Let $X$ be centered square integrable process with stationary increments. Then the increments $Y_k := X_k - X_{k-1}$ are stationary with autocorrelation function

$$\rho(k) = \frac{1}{2} \left( (k+1)^{2H} - 2k^{2H} + (k-1)^{2H} \right).$$

Therefore, as $k$ tends to infinity one has

$$\rho(k) \approx H(2H-1) k^{2H-2},$$

i.e.

$$\lim_{k \to \infty} \frac{\rho(k)}{H(2H-1)k^{2H-2}} = 1.$$

Thus, if $H \in (\frac{1}{2}, 1)$ the increments $(Y_k)_{k \in \mathbb{N}}$ exhibit the long-range dependency property with $\alpha = 2 - 2H$ and $c_\rho = H(2H-1)$.

Note that by Kolmogorov's criterion the process $X$ admits a version with continuous sample paths in the case $H \in (\frac{1}{2}, 1)$.

Finally, one should note that there are centered processes with stationary and *independent* increments having the statistical self-similarity property with index $H > \frac{1}{2}$, e.g. the symmetric $\alpha$-stable Lévy processes with $H = \frac{1}{\alpha}$. These are processes with infinite variance, however. For details we refer to Samorodnitsky and Taqqu [22]

## 3. Fractional Brownian motion

The fractional Brownian motion $Z = Z^H$ is a continuous and centered Gaussian process with the covariance function

$$\mathbf{E}Z_t Z_s = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

This is a valid covariance function for a Gaussian process (cf. Samorodnitsky and Taqqu [22, p. 106]). So, by virtue of (2.4) the fractional Brownian motion is the (upto a multiplicative constant) unique centered Gaussian process with stationary increments having the self-similarity property with Hurst index $H \in (0,1)$.

The fractional Brownian motion was originally defined and studied by Kolmogorov [11] within a Hilbert space framework where it was called a Wiener helix. It was further studied by Yaglom [27]. The name "fractional Brownian motion" comes from Mandelbrot and Van Ness [15]. They defined it as a stochastic integral with respect to the standard Brownian motion:

$$Z_t = \int_{-\infty}^{t} k(t,s)\,\mathrm{d}W_s \tag{3.1}$$

with a certain deterministic kernel $k$ depending on $H$.

If $H = \frac{1}{2}$ then the fractional Brownian motion is just a standard Brownian motion. In the case $H > \frac{1}{2}$ the fractional Brownian motion exhibits the statistical long-range dependency property as shown in the previous section. Hence, in financial modeling one usually assumes that $H \in (\frac{1}{2}, 1)$ (see, however, Shiryaev [24, p. 347] and references therein).

In addition to the Mandelbrot and Van Ness integral representation (3.1) of the fractional Brownian motion there exists a similar representation where the integration is taken over the finite interval $[0,t]$, viz.

$$Z_t = \int_0^t z(t,s)\,\mathrm{d}W_s, \tag{3.2}$$

where

$$z(t,s) = c_1\left[\left(\tfrac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\tfrac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u\right].$$

The $c_1$ is a certain normalizing constant depending on $H$. The representation (3.2) is canonical in the sense that the filtrations generated by $Z$ and $W$ coincide. For the proof of the representation see Norros, Valkeila and Virtamo [18].

So far everything is true for $H \in (0,1)$. For the rest of the paper we assume that $H > \frac{1}{2}$, however (although some results are true for the case $H \in (0,\frac{1}{2})$ also).

To define a fractional analogue of the classical Black & Scholes pricing model we need to know how to integrate with respect to the fractional Brownian motion as this connected to hedging. Also, we need to have the Girsanov theorem for the fractional Brownian motion to consider an analogue of the equivalent martingale measure. The rest of the section is dedicated to these technical issues.

3.1. **$p$-variation and integration.** To define a (pathwise) stochastic integral with respect to the fractional Brownian motion we consider the so-called $p$-variation index of a stochastic process. For details of this approach we refer to Dudley and Norvaiša [4].

Let $\kappa = (0 = t_0 < t_1 < \cdots < t_n = T)$ be a finite partition of the interval $[0,T]$ and for a stochastic process $X$ set

$$s_p(X,\kappa) \ := \ \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p.$$

The *$p$-variation* $v_p$ of a stochastic process $X$ over an interval $[0,T]$ is defined as

$$v_p(X,[0,T]) \ := \ \sup_{\kappa} s_p(X,\kappa)$$

where $\kappa$ is a finite partition of the interval $[0,T]$. The *index of $p$-variation* $v$ of a process $X$ is then

$$v(X,[0,T]) \ = \ \inf\{p > 0 : v_p(X,[0,T]) < \infty\}$$

if the set above is non-empty and $\infty$ otherwise.

For fractional Brownian motion $Z$ with index $H \in (\frac{1}{2},1)$ one has

$$v(Z) \ = \ v(Z,[0,T]) \ = \ \frac{1}{H}. \tag{3.3}$$

For the proof we refer to Dudley and Norvaiša [4, p. 48].

For semimartingales $M$ one must have $v(M) \in [0,1] \cup \{2\}$ (cf. Dudley and Norvaiša [4, p. 46]). Thus, the fractional Brownian motion is not a semimartingale when $H \neq \frac{1}{2}$. So, one cannot use the Ito theory to define stochastic integrals with respect to it. However, one can define a pathwise, i.e. $\omega$-by-$\omega$, integrals as a refinement of the Riemann–Stieltjes integrals using the $p$-variation.

The following change of variables formula for the pathwise integration with respect to the fractional Brownian motion plays a central role in what follows.

Suppose that $f \in C^1([0,T])$ then

$$f(Z_t) - f(Z_s) \ = \ \int_s^t f'(Z_u)\,\mathrm{d}Z_u. \tag{3.4}$$

Note the absence of the "Ito correction term", or the quadratic variation term. Also, one should note that the formula (3.4) is valid in the all the different pathwise approaches to define the stochastic integral (cf. Lin [13], Föllmer [6], Zähle [28]).

3.2. **Girsanov theorem.** We recall the Girsanov theorem for fractional Brownian motion. For details we refer to Norros, Valkeila and Virtamo [18] and Molchan [17].

Define the so-called fundamental martingale $M = M^H$ by

$$M_t \;:=\; c_2 \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\,\mathrm{d}Z_s$$

Now $M$ is a Gaussian martingale with the angle bracket

$$\langle M\rangle_t \;=\; c_3 t^{2-2H}.$$

Here $c_2$ and $c_3$ are certain constants depending on $H$.

Let $\rho$ be a deterministic function and define a measure $\mathbf{Q} = \mathbf{Q}^\rho$ by

$$\begin{aligned}
\text{(3.5)}\quad \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\Big|\mathscr{F}_t^Z \;&=\; \exp\left(\int_0^t \rho(s)\,\mathrm{d}M_s - \frac{1}{2}\int_0^t \rho^2(s)\,\mathrm{d}\langle M\rangle_s\right) \\
&=\; \exp\left(\int_0^t \rho(s)\,\mathrm{d}M_s - \frac{1}{2}c_3(2-2H)\int_0^t \rho^2(s)s^{1-2H}\,\mathrm{d}s\right).
\end{aligned}$$

Then the process

$$Z - \int_0^\cdot \rho(t)\,\mathrm{d}t$$

is a fractional Brownian motion under $\mathbf{Q}$ and under $\mathbf{Q}$ only (given that $\mathbf{Q}$ is equivalent to $\mathbf{P}$ ).

3.3. **Prediction.** We have the following prediction formula for the fractional Brownian motion:

$$\hat{Z}_{T|t} \;:=\; \mathbf{E}\big[Z_T|\mathscr{F}_t^Z\big] \;=\; Z_t + \int_0^t \Psi_T(t,s)\,\mathrm{d}Z_s, \tag{3.6}$$

where

$$\Psi_T(t,s) \;=\; \frac{\sin(\pi(H-\frac{1}{2}))}{\pi} s^{\frac{1}{2}-H}\int_t^T \frac{u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}}{u-s}\,\mathrm{d}u.$$

The formula (3.6) is due to Gripenberg and Norros [7].

There is also an alternative form

$$\hat{Z}_{T|t} \;=\; \int_0^t z(T,s)\,\mathrm{d}W_s. \tag{3.7}$$

This is a direct consequence of (3.2). It provides us the formula for the bracket of $\hat{Z}_{T|\cdot}$ needed later:

$$\langle \hat{Z}_{T|\cdot}\rangle_t \;=\; \int_0^t z(T,s)^2\,\mathrm{d}s.$$

Since the function $z(T,\cdot)$ is positive on $(0,T)$ for all $T>0$ the relation (3.7) can be inverted. This provides us the following representation for the Brownian motion $W$:

$$W_t = \int_0^t \frac{1}{z\,(T,s)}\,\mathrm{d}\hat{Z}_{T|s},$$

for $t \le T$. This shows that

$$\mathscr{F}_t^Z = \mathscr{F}_t^W = \mathscr{F}_t^{\hat{Z}_{T|\cdot}} \tag{3.8}$$

for $0 \le t \le T$.

## 4. Fractional pricing model

Let us modify the classical Black & Scholes model. As a source of randomness replace the Brownian motion $W$ with the fractional one $Z$ with index $H \in (\frac{1}{2}, 1)$ and consider the following dynamics for the stock price $S$:

$$\mathrm{d}S_t = S_t\,(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t)\,. \tag{4.1}$$

The solution to the stochastic differential equation (4.1) is called the geometric fractional Brownian motion.

So, the fractional model adds one parameter, the Hurst index $H$, to the classical Black & Scholes model to capture the long-range dependence of the log-returns of the stock. As in the classical case one assumes that the values of the parameters $r$, $\mu$, $\sigma$ and $H$ are known.

Since, by virtue of the property (3.3), the fractional Brownian motion is not a semimartingale when $H \neq \frac{1}{2}$ we are faced with the following problems.

(a) How to define the stochastic integral (4.1)?
(b) Is the modified pricing model free of arbitrage?
(c) Is the modified pricing model complete, i.e. is there a fractional analogue of the Ito–Clark–Ocone formula?

For the problem (a) there are two possible definitions, viz. the *pathwise* definition introduced earlier and a definition based on *generalized stochastic* processes.

The next section considers briefly the generalized solutions. The rest of the paper considers the pathwise solutions.

### 4.1. Generalized geometric fractional Brownian motion.

The generalized solution to (4.1) is based on white noise analysis and the so-called Wick products (cf. Duncan, Hu and Pasic-Duncan [5] and for financial application Hu and Øksendal [9]). The solution is

$$S_t = S_0 \exp\left(\mu t - \frac{\sigma^2}{2} t^{2H} + \sigma Z_t\right).$$

With generalized solutions the fractional pricing model is free of arbitrage and complete. Thus, the problems (b) and (c) are solved.

Although this formal result is satisfactory from the theoretical point of view, there are some problems. Firstly, the functional analytic approach used makes it impossible to consider the integrals as almost sure limits of the paths of the process under certain partitions of the interval. It should be noted that this interpretation is possible in the case of Brownian motion. Secondly, and more subjectively, one wants to model the path properties of the stock price. The generalized approach does not fit well to this aim since the path properties play no central role in the integrals.

4.2. **Arbitrage in fractional models.** By the change of variables formula (3.4) the pathwise solution to (4.1) is

$$S_t = S_0 \exp(\mu t + \sigma Z_t)\,.$$

With pathwise solutions to (4.1) and with *continuous-time* trading one can do arbitrage in the fractional pricing model.

Probably the first one to show that there exist arbitrage opportunities with fractional Brownian motion (although he did not consider the *geometric* fractional Brownian motion but a linear one) was Rogers [20]. His strategy consisted of combinations of buy and hold strategies. The construction was rather implicit and relied heavily on the self-similarity of fractional Brownian motion and information from the whole interval $(-\infty, t]$.

Shiryaev [23] (see also Dasgupta [3]) constructed an arbitrage opportunity explicitly by using the notion of pathwise stochastic integral. He considered the case $\mu = r$, $\sigma = 1$ and constructed a portfolio $\pi = (\pi(B), \pi(S))$ with

$$\begin{aligned} \pi(B)_t &= 1 - e^{2Z_t}, \\ \pi(S)_t &= 2(e^{Z_t} - 1). \end{aligned}$$

The associated value process $V$ satisfies

$$\mathrm{d}V_t = 2e^{rt+Z_t}\left(e^{Z_t} - 1\right)\mathrm{d}Z_t + \left(e^{Z_t} - 1\right)^2 re^{rt}\,\mathrm{d}t$$

as easily seen by the change of variables formula (3.4). Hence, the portfolio is self-financing. There is arbitrage here, however. To see this just note that

$$V_T = e^{rT}\left(e^{Z_T} - 1\right)^2.$$

In contrast to the arbitrage portfolio (1.4) constructed for the classical model the portfolio $\pi$ is admissible in the sense that it is non-negative (there can be no martingale characterization here).

Recently, Cheridito [2] has constructed an arbitrage opportunity by using the $\frac{1}{H}$-variation property (3.3) of the paths of the fractional Brownian motion. The construction was not based on continuous trading but on a finite number of trading points $t_1, \ldots, t_n \in [0, T]$. The number of the points depend on the path, however. The value process of his portfolio was bounded from below by an arbitrary negative constant (i.e. it was tame). It is worthwhile to notice that Cheridito's construction was based solely on the fact that the $p$-variation of the paths is not 2. Hence it goes beyond the fractional Brownian motion.

All the arbitrage opportunities mentioned above, save that of Rogers' [20], apply also to the case where the source of randomness is a process with bounded variation.

Let us note that the arbitrage arises in a modified binomial approximation. Indeed, construct a weighted random walk $Z^{(n)}$ using the kernel $z$ in (3.2) by

$$Z_t^{(n)} \ := \ \int_0^t z^{(n)}(t,s)\,\mathrm{d}W_s^{(n)} \ := \ \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt \rfloor}{n}, s)\,\mathrm{d}s\,\frac{1}{\sqrt{n}}\xi_i^{(n)},$$

where the $\xi_i^{(n)}$'s are independent random variables with

$$\mathbf{P}\left(\xi_i^{(n)} = 1\right) \ = \ \frac{1}{2} \ = \ \mathbf{P}\left(\xi_i^{(n)} = -1\right)$$

and

$$z^{(n)}(t,s) \ = \ n\int_{s-\frac{1}{n}}^{s} z(\tfrac{\lfloor nt \rfloor}{n}, u)\,\mathrm{d}u.$$

is a piecewise constant approximation to $z(t,\cdot)$. Now, if

$$\Delta S_k^{(n)} \ := \ S_{k-1}^{(n)}\left(\mu\frac{T}{n} + \sigma \Delta Z_{k\frac{T}{n}}^{(n)}\right)$$

is the stock price dynamics at time $k\frac{T}{n}$ one obtains an analogue of the Cox–Ross–Rubinstein binomial approximation to Black & Scholes model. This fractional binomial, or binary, model exhibits arbitrage opportunities given that the level of approximation, $\frac{T}{n}$, is big enough (depending on $H$, but finite). For details and explicit construction of arbitrage we refer to Sottinen [25].

So, the classical notion of admissibility is not enough to exclude arbitrage opportunities in the fractional pricing model. If one restricts trading so that there must be some non-random minimal time interval between successive transactions (which may be random) then there is no arbitrage (cf. Cheridito [2]). This is very restrictive, however. Indeed, there are many claims which cannot be replicated with these strategies. At the present moment it is not clear what should be the notion of admissibility in the fractional pricing model. For some discussion (and yet another construction of arbitrage) see Salopek [21].

4.3. **European options in fractional models.** In spite of the lack of the equivalent martingale measure one can compute the prices of European options in the pathwise fractional model using a so-called weak pricing principle (cf. Valkeila [26]). These prices coincide with the ones obtained in the generalized model (cf. Hu and Øksendal [9]).

The risk neutral measure $\mathbf{Q}$, equivalent to the real world measure $\mathbf{P}$, in the classical Black & Scholes pricing model was characterized by the fact that the discounted stock price $S/B$ is a martingale. In particular, the average growth of the stock is the growth of the bond under $\mathbf{Q}$ :

$$\mathbf{E}_{\mathbf{Q}}\frac{S_t}{B_t} \ = \ S_0. \tag{4.2}$$

In the fractional setting we cannot hope to have the martingale property. However, the Girsanov theorem for fractional Brownian motion provides us a unique probability measure, equivalent to $\mathbf{P}$, such that (4.2) holds. We shall call that measure $\mathbf{Q}$ the *average risk neutral measure.*

The equation (4.2) holds if the process

$$(t,\omega) \;\mapsto\; Z_t(\omega) + \frac{\mu - r}{\sigma} t + \frac{\sigma}{2} t^{2H} \tag{4.3}$$

is the fractional Brownian motion under $\mathbf{Q}$. By the Girsanov theorem the average risk neutral measure $\mathbf{Q}$ is therefore characterized by

$$\begin{aligned} \log \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\Big|\mathscr{F}_t^Z \;&=\; -\frac{\mu - r}{\sigma} M_t + \sigma H \int_0^t s^{2H-1}\,\mathrm{d}M_s \\ &\quad -\frac{1}{2} c_3(2-2H)\left(\left(\frac{\mu - r}{\sigma}\right)^2 t^{2-2H} + (\mu - r)t + \sigma^2 t^{2H}\right). \end{aligned}$$

The discounted stock price process $S/B$ is not a $\mathbf{Q}$-martingale. Indeed, by using the prediction theorem (3.6) we can calculate conditional expectations under the measure $\mathbf{Q}$ :

$$\mathbf{E}_{\mathbf{Q}}\left[\frac{S_T}{B_T}\Big|\mathscr{F}_t^Z\right] \;=\; \frac{S_t}{B_t} e^{K(T,t)},$$

where

$$\begin{aligned} K(T,t) \;&=\; (r-\mu)\int_0^t \Psi_T(t,s)\,\mathrm{d}s \\ &\quad + \frac{\sigma^2}{2}\left(t^{2H} - \int_0^t z(T,s)^2\,\mathrm{d}s\right) \\ &\quad + \sigma \int_0^t \Psi_T(t,s)\,\mathrm{d}Z_s \\ &\quad - \sigma^2 \int_0^t s^{2H-1}\Psi_T(t,s)\,\mathrm{d}s. \end{aligned}$$

This is consistent with the classical case with Brownian motion $W$ in the sense that $K = K_H \to 0$ as $H \downarrow \frac{1}{2}$.

The computation of the conditional expectations is based on the following identity:

$$S_t \;=\; S_0 \exp\left(\mu T - \frac{\sigma^2}{2} T^{2H}\right) \mathscr{E}(\sigma \hat{Z}_{T|\cdot})_T, \tag{4.4}$$

where

$$\mathscr{E}\left(\sigma \hat{Z}_{T|\cdot}\right)_t \;:=\; \exp\left(\sigma \hat{Z}_{T|t} - \frac{\sigma^2}{2}\int_0^t z\,(T,s)^2\,ds\right)$$

is the stochastic exponential.

For a European option $f = f(S_T)$ we then define the *average expectation price* $C_T(f)$ to be

$$C_T(f) \;:=\; \mathbf{E}_\mathbf{Q}\frac{f(S_T)}{B_T}. \tag{4.5}$$

This is analogous to the classical Black & Scholes case, i.e. formula (1.1). As in the classical Black & Scholes model we want to give an expression to the discounted value process $V/B$. So, we look at

$$\mathbf{E}_\mathbf{Q}\left[\frac{f(S_T)}{B_T}\Big|\mathscr{F}_t^S\right] \;=\; \mathbf{E}_\mathbf{Q}\left[\frac{V_T}{B_T}\Big|\mathscr{F}_t^S\right] \;=\; \mathbf{E}_\mathbf{Q}\left[\frac{V_T}{B_T}\Big|\mathscr{F}_t^{\hat{S}_{T|\cdot}}\right],$$

where

$$\hat{S}_{T|t} \;=\; \mathscr{E}\big(\sigma \hat{Z}_{T|\cdot}\big)_t.$$

Note that the change of filtration is justified by (3.8).

It seems possible to calculate the hedging portfolios with $\hat{S}_{T|\cdot}$ explicitly as non-anticipative functionals of the path of the stock price process $S$. The strategies will not be of the Markov type but rather of the form

$$\pi_t \;=\; K\left(T, t, (S_s)_{s\in[0,t]}\right).$$

The calculations are expected to be rather tedious, however.

Since (4.3) is a fractional Brownian motion under $\mathbf{Q}$ we can express the formula (4.5) as

$$C_T(f) \;=\; e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\left(S_0\exp\!\left(\sigma y T^H + rT - \frac{\sigma^2}{2}T^{2H}\right)\right)e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

Finally, let us state the celebrated Black & Scholes formula for a European call-option with strike price $K$ in the fractional case:

$$C_T((S_T-K)^+) \;=\; S_0\Phi(y_1) - Ke^{-rT}\Phi(y_2),$$

where

$$y_1 \;=\; y_1(H) \;=\; \frac{\log\frac{S_0}{K} + rT + \frac{\sigma^2}{2}T^{2H}}{\sigma T^H},$$

$$y_2 \;=\; y_2(H) \;=\; \frac{\log\frac{S_0}{K} + rT - \frac{\sigma^2}{2}T^{2H}}{\sigma T^H}.$$

Note that the price converges to the classical Black & Scholes price for European call-option as $H \downarrow \frac{1}{2}$.

## 5. Beyond fractional pricing model

To overcome the shortcomings of the fractional pricing model some remedies have been proposed. We shall briefly consider two approaches, viz. regularization procedures and mixed models.

5.1. **Regularization.** The regularization procedure to the fractional Brownian motion was suggested by Rogers [20] and was further studied by Cheridito [2]. For details we refer to the latter.

Recall the Mandelbrot and Van Ness integral representation (3.1) for the fractional Brownian motion. One can regularize the fractional Brownian motion in the following way. Replace the kernel $k$ in (3.1) by a "regularized" kernel $\tilde{k}$ so that the Gaussian process

$$\tilde{Z}_t \;:=\; \int_{-\infty}^{t} \tilde{k}(t,s)\,\mathrm{d}W_s$$

has stationary increments and satisfies the following two objectives.

(a) The process $\tilde{Z}$ is close to the original fractional Brownian motion $Z$ in the sense that

$$\mathbf{Cov}(Z_t, Z_s) \;\approx\; \mathbf{Cov}(\tilde{Z}_t, \tilde{Z}_s)$$

for all $t, s \in [0,T]$.

(b) The law of the process $\tilde{Z}$ is equivalent to the law of the standard Brownian motion.

If the objective (a) is satisfied then the process $\tilde{Z}$ is a reasonable replacement to the fractional Brownian motion from the statistical point of view. The objective (b) gives us the unique risk neutral, or pricing, measure $\mathbf{Q}$ of the classical Black & Scholes pricing model. In particular, by the Hitsuda representation theorem (cf. Hitsuda [8]), the objective (b) is satisfied (under the assumption of stationary increments) if and only if

$$\tilde{Z}_t \;=\; W_t + \int_{-\infty}^{t}\int_{-\infty}^{s} \psi(s-u)\,\mathrm{d}W_u\,\mathrm{d}s$$

for some square integrable kernel $\psi$.

While it is possible to construct suitable kernels $\psi = \psi_\varepsilon$ such that the objective (a) is satisfied with any precision $\varepsilon > 0$ there is a problem. The option prices introduced by the kernel $\psi$ depend heavily on the form of the kernel while the precision $\varepsilon$ does not play a particularly significant role. It seems difficult to argue any specific form for the kernel $\psi$, however.

5.2. **Mixed models.** Consider a mixed Brownian–Fractional Brownian model for the stock price, viz.

$$\mathrm{d}S_t \;=\; S_t\left(\mu\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t + \sigma\,\mathrm{d}Z_t\right), \tag{5.1}$$

where $W$ is a standard Brownian motion and $Z$ is a fractional one.

Cheridito [2, p. 73] has shown that the mixed process $\varepsilon W + \sigma Z$ is equivalent in law to $\varepsilon W$ whenever $W$ and $Z$ are independent and the index of self-similarity $H$ satisfies $H \in (\frac{3}{4}, 1)$. Hence, this model is similar to the regularized one considered in the previous subsection in the sense that we can now make use of the unique risk neutral measure $\mathbf{Q}$. Also, one can make the process $\varepsilon W + \sigma Z$ to be as close as one wants to the fractional Brownian motion $\sigma Z$ in the statistical sense by choosing a small enough $\varepsilon$.

Mishura and Valkeila [16] also considered a mixed model of the type (5.1). The standard Brownian motion and the fractional one were not independent in their case, however. Indeed, the fractional Brownian motion was constructed from the standard Brownian motion by the formula

$$Z_t = ct^{H-\frac{1}{2}} \int_0^t (t-s)^{H-\frac{3}{2}} M_s \,\mathrm{d}s - c' \int_0^t s^{H-\frac{3}{2}} \int_0^s (s-u)^{H-\frac{3}{2}} M_u \,\mathrm{d}u \,\mathrm{d}s,$$

where $c$ and $c'$ are certain constants depending on $H$ and

$$M_t = \int_0^t s^{\frac{1}{2}-H} \,\mathrm{d}W_s.$$

They showed that there is no arbitrage in this model within the class on *Markov* strategies. Although this seems to be a very weak form of no-arbitrage note that the arbitrage opportunities constructed for the fractional pricing model [2, 3, 21, 23] were all of the Markov type.

## Acknowledgements

Sottinen was financed by the Finnish Graduate School in Stochastics.

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Tommi Sottinen, Department of Mathematics, P.O. Box 4, 00014 University of Helsinki, Finland

*E-mail address*: `tommi.sottinen@helsinki.fi`

Esko Valkeila, Department of Mathematics, P.O. Box 4, 00014 University of Helsinki, Finland

*E-mail address*: `esko.valkeila@helsinki.fi`