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On arbitrage and replication in the fractional Black–Scholes pricing model

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Summary: It has been proposed that the arbitrage possibility in the fractional Black–Scholes model depends on the definition of the stochastic integral. More precisely, if one uses the Wick–Itô–Skorohod integral one obtains an arbitrage-free model. However, this integral does not allow economical interpretation. On the other hand it is easy to give arbitrage examples in continuous time trading with self-financing strategies, if one uses the Riemann-Stieltjes integral. In this note we discuss the connection between two different notions of self-financing portfolios in the fractional Black–Scholes model by applying the known connection between these two integrals. In particular, we give an economical interpretation of the proposed arbitrage-free model in terms of Riemann–Stieltjes integrals.

1 Introduction

There are some studies of financial time series indicating that the stock market prices exhibit the so-called long range dependency property. For references to these studies see e.g. Shiryaev [18]. Therefore, it has been proposed that one should replace the Brownian motion (which has no memory) in the classical Black–Scholes pricing model by a process with long memory. A simple modification is to introduce fractional Brownian motion as the source of randomness. Thus one adds one parameter, H, to model the dependence structure in the stock prices. It should be noted that the fractional Brownian motion is a Gaussian process and there are statistical studies indicating that the log-returns are not Gaussian. We ignore this point completely in what follows. The fractional Brownian motion $Z = Z^H$ with index $H \in (0, 1)$ is a continuous and centered Gaussian process with the covariance function

$$R(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

The fractional Brownian motion was introduced by Kolmogorov [9] under the name “Wiener Spiral”. The current name comes from the other pioneering paper by Mandelbrot and Van Ness [10]. We shall assume that $H \in (\frac{1}{2}, 1)$ as this is the case when the fractional

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Brownian motion has the long range dependency property. In this case the covariance function R is of bounded variation and it can be written as the double integral

$$R(t,s) \;=\; H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v. \tag{1.1}$$

If $H=\frac{1}{2}$, then the fractional Brownian motion is standard Brownian motion, but if $H\neq\frac{1}{2}$ the fractional Brownian motion is not a semimartingale. Hence, one cannot use the martingale theory to define stochastic integrals with respect to it. Various authors have proposed basically two types of integrals; pathwise and divergence integrals. In this paper we study the connection between these two different notions as they are used in connection to a fractional Black–Scholes model. In particular, we study the two self-financing conditions corresponding to the different integrals.

This paper is organized as follows. The pathwise Riemann–Stieltjes integration with respect to fractional Brownian motion is introduced in Section 2. In Section 3 we recall some preliminaries of Malliavin calculus and introduce the Wick–Itô–Skorohod integral as (a restriction of) the Malliavin divergence. The fractional Black–Scholes pricing model is introduced in Section 4. Finally, in Section 5 we consider the implications of different notions of integrals to the problem of arbitrage and self-financing condition in the fractional pricing model.

2 Pathwise integrals with respect to fractional Brownian motion

The fractional Brownian motion has almost surely sample paths of unbounded variation. Nevertheless, one can define Riemann–Stieltjes integrals with respect to it if one assumes that the integrand has some smoothness properties. Of the various ways of doing this we recall the so-called p-variation approach introduced by Young [22]. For more details see Dudley and Norvaiša [6].

Consider partitions $\pi := \{t_k\,:\,0=t_0<t_1<\cdots<t_n=T\}$ of the interval $[0,T]$. For a function f on $[0,T]$ set

$$v_p(f;\pi) \;:=\; \sum_{t_k\in\pi} |f(t_k)-f(t_{k-1})|^p\,.$$

If

$$v_p(f) \;:=\; \sup_\pi v_p(f;\pi)$$

is finite we say that f has *bounded p-variation* and denote $f\in\mathscr{W}_p$.

Young [22] proved that if $f\in\mathscr{W}_p$ and $g\in\mathscr{W}_q$ for some p and q satisfying $1/p+1/q>1$ and have no common discontinuities then the integral

$$\int_0^T g(t)\,\mathrm{d}f(t)$$

exists in the Riemann–Stieltjes sense (i.e. as a limit of Riemann–Stieltjes sums).
It can be shown that the paths of the fractional Brownian motion Z with index H belong almost surely to the space $\mathscr{W}_p$ if (and only if) $p > 1/H$, cf. Dudley and Norvaiša [6]. Consequently, the *Riemann–Stieltjes integral*

$$\int_0^1 u_t \,\mathrm{d}Z_t$$

exists almost surely if $u \in \mathscr{W}_q$ for some $q < 1/(1-H)$ almost surely.
Since the pathwise integral is a Riemann–Stieltjes integral we have the classical change of variables formula, or the Itô formula, valid for fractional Brownian motion Z with $H > \frac{1}{2}$.

Proposition 2.1 *Suppose that $F \in C^{1,1}([0,T] \times \mathbb{R})$. Then the equation*

$$F(t, Z_t) - F(s, Z_s) \;=\; \int_s^t \frac{\partial F}{\partial t}(u, Z_u)\,\mathrm{d}u + \int_s^t \frac{\partial F}{\partial x}(u, Z_u)\,\mathrm{d}Z_u \tag{2.1}$$

holds almost surely for all $s, t \in [0,T]$.

3 Wick–Itô–Skorohod integrals with respect to fractional Brownian motion

The study of Skorohod integration with respect to the fractional Brownian motion was initiated by Decreusefond and Üstünel [5] and Duncan et al. [7]. In [5] the theory was developed by using Malliavin calculus. In [7] the authors used generalised stochastic processes and Wick calculus. The use of Skorohod integration in connection to finance was proposed by Hu and Øksendal [8].
We recall some preliminaries of Malliavin calculus to define the Skorohod integral with respect to fractional Brownian motion with $H > \frac{1}{2}$. For details of Malliavin calculus in general we refer to Nualart [14] and in the special case of fractional Brownian motion to Alòs et al. [1, 2].
Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space where the σ-algebra $\mathscr{F}$ is generated by the fractional Brownian motion.
Denote by $\mathscr{H}$ the linear span of the indicator functions $\{\mathbf{1}_{[0,t]} : t \in [0,T]\}$ (i.e. the step functions) completed with respect to the inner product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathscr{H}} := R(t,s).$$

So, if f and g are step functions then by (1.1) we have

$$\langle f, g\rangle_{\mathscr{H}} \;=\; H(2H-1)\int_0^T\int_0^T f(t)g(s)|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t.$$

The mapping $\mathbf{1}_{[0,t]} \mapsto Z_t$ extends to an isometry between $\mathscr{H}$ and $\mathscr{H}_1$, the linear space of Z. We denote by $Z(\varphi)$ the image of $\varphi \in \mathscr{H}$ in this isometry. The Hilbert space $\mathscr{H}$ is not a function space but contains distributions of negative type. Therefore, we introduce $|\mathscr{H}| \subset \mathscr{H}$, a linear space of measurable functions f on $[0,T]$ such that

$$\|f\|^2_{|\mathscr{H}|} \;:=\; H(2H-1)\int_0^T\int_0^T |f(t)f(s)||t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \;<\; \infty.$$

Similarly, $|\mathscr{H}|^{\otimes 2} \subset \mathscr{H}^{\otimes 2}$ is the space of measurable functions on $[0,T]^2$ satisfying

$$\|f\|^2_{|\mathscr{H}|^{\otimes 2}} \;:=\; H^2(2H-1)^2 \int\limits_{[0,T]^4} |f(r,s)f(t,u)|\,(|r-t||s-u|)^{2H-2}\;\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u \;<\; \infty.$$

Let $\mathscr{S}$ be the collection of random variables of the form

$$F \;=\; f\left(Z(\varphi_1),\ldots,Z(\varphi_n)\right)$$

for some $n \geq 1$, $\varphi_k \in \mathscr{H}$, $k=1,\ldots,n$, and $f \in C_b^\infty(\mathbb{R}^n)$, i.e. f and all its derivatives are bounded.

For $F \in \mathscr{S}$ the *Malliavin derivative* of F is the $\mathscr{H}$-valued random variable

$$DF \;:=\; \sum_{k=1}^n \frac{\partial f}{\partial x_k}\left(Z(\varphi_1),\ldots,Z(\varphi_n)\right)\varphi_k.$$

For any $p \geq 1$ the derivative operator D is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega;\mathscr{H})$. Similarly, the k times iterated operator D^k maps $L^p(\Omega)$ into $L^p(\Omega;\mathscr{H}^{\otimes k})$. The Sobolev space $\mathbb{D}^{k,p}$, the *domain* of D^k in $L^p(\Omega)$, is the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|^p_{k,p} \;:=\; \mathbf{E}\left(|F|^p + \sum_{j=1}^k \|D^jF\|^p_{\mathscr{H}^{\otimes j}}\right).$$

In a similar way, given a Hilbert space V we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The divergence operation δ is introduced as the adjoint of D. The *domain* of δ, denoted by $\mathrm{Dom}\,\delta$, is the set of $\mathscr{H}$-valued random variables u satisfying

$$\left|\mathbf{E}\left\langle DF,u\right\rangle_{\mathscr{H}}\right|^2 \;\leq\; c\mathbf{E}F^2$$

for all $F \in \mathscr{S}$. Here c is a constant depending on u.

For $u \in \mathrm{Dom}\,\delta$ the *divergence* $\delta(u)$ is a square integrable random variable defined by the duality relation

$$\mathbf{E}\delta(u)F \;=\; \mathbf{E}\left\langle DF,u\right\rangle_{\mathscr{H}} \tag{3.1}$$

for all "test variables" $F \in \mathbb{D}^{1,2}$.

For any $p > 1$ we denote by $\mathbb{D}^{1,p}(|\mathscr{H}|)$ the subspace of $\mathbb{D}^{1,p}(\mathscr{H})$ consisting of elements u satisfying $u \in |\mathscr{H}|$, $Du \in |\mathscr{H}|^{\otimes 2}$ and

$$\mathbf{E}\Big(\|u\|^p_{|\mathscr{H}|} + \|Du\|^p_{|\mathscr{H}|^{\otimes 2}}\Big) \;<\; \infty.$$

Note that $\mathbb{D}^{1,2}(|\mathscr{H}|) \subset \mathbb{D}^{1,2}(\mathscr{H}) \subset \mathrm{Dom}\,\delta$.
The *Wick–Itô–Skorohod integral* of a process $u \in \mathbb{D}^{1,2}(|\mathscr{H}|)$ with respect to fractional Brownian motion Z is just the divergence, i.e.

$$\int_0^T u_t\,\delta Z_t \;\; := \;\; \delta(u).$$

The reason to call this Itô–Skorohod integral is that in the case of Brownian motion the divergence coincides with the extension of the Itô integral introduced by Skorohod. Moreover, if one introduces a so-called Wick product $F \diamond G$ one can show that under some regularity assumptions on u we have

$$\int_0^T u_t\,\delta Z_t \;\; = \;\; \lim_{|\pi|\to 0} \sum_{t_k\in\pi} u_{t_{k-1}} \diamond (Z_{t_k} - Z_{t_{k-1}}). \tag{3.2}$$

Here the convergence is the $L^2(\Omega)$-convergence of random variables (see Alòs et al. [1] for details on the convergence result (3.2) and Bender [3] on the Wick product). So, the Wick–Itô–Skorohod integral can be considered as a limit of Riemann–Stieltjes sums if one replaces the ordinary product by the Wick product.
The following form of the Itô formula for the Wick–Itô–Skorohod integrals can be found in Bender [3]. Actually, in [3] the Wick–Itô–Skorohod integral was defined by using the white noise analysis instead of Malliavin calculus.

Proposition 3.1 *Let* $F \in C^{1,2}([0,T]\times\mathbb{R})$ *satisfy the growth condition*

$$\max\left(\Big|F(t,x)\Big|, \Big|\frac{\partial F}{\partial t}(t,x)\Big|, \Big|\frac{\partial F}{\partial x}(t,x)\Big|, \Big|\frac{\partial^2 F}{\partial x^2}(t,x)\Big|\right) \;\; \le \;\; Ce^{\lambda x^2}$$

where C *and* λ *are positive constants and* $4T^{2H}\lambda < 1$. *Let* $s,t \in [0,T]$. *Then the equation*

$$\begin{aligned} F(t,Z_t) - F(s,Z_s) \;\; &= \;\; \int_s^t \frac{\partial F}{\partial x}(u,Z_u)\,\delta Z_u + \int_s^t \frac{\partial F}{\partial t}(u,Z_u)\,\mathrm{d}u \\ &\quad + H\int_s^t \frac{\partial^2 F}{\partial x^2}(u,Z_u)u^{2H-1}\,\mathrm{d}u \end{aligned} \tag{3.3}$$

holds in $L^2(\Omega)$.

4 Two approaches to pricing model

In the classical *Black–Scholes model* the (discounted) stock price process S is given by the Itô differential equation

$$\mathrm{d}S_t \;\; = \;\; S_t\mu(t)\,\mathrm{d}t + S_t\sigma\,\mathrm{d}\tilde{W}_t$$

where $\tilde{W}$ is a Brownian motion under the real world measure $\mathbf{P}$ and μ is a deterministic function. Under the so-called equivalent martingale measure $\mathbf{Q}$ we have, taking $\sigma = 1$,

$$\mathrm{d}S_t \;\; = \;\; S_t\,\mathrm{d}W_t.$$

Here W is a $\mathbf{Q}$-Brownian motion and the change of measure is given by the well-known Girsanov formula for Brownian motion. In particular, the stock price process is a geometric Brownian motion (and hence a martingale) under $\mathbf{Q}$, i.e.

$$S_t = S_0 e^{W_t - \frac{1}{2}t}. \tag{4.1}$$

In the *fractional Black–Scholes model* the (discounted) stock price is given by either the Riemann–Stieltjes equation

$$\mathrm{d}S_t = S_t \mu(t)\,\mathrm{d}t + S_t \sigma\,\mathrm{d}\tilde{Z}_t \tag{4.2}$$

or the Wick–Itô–Skorohod equation

$$\delta S_t = S_t \nu(t)\,\mathrm{d}t + S_t \sigma\,\delta\tilde{Z}_t \tag{4.3}$$

where $\tilde{Z}$ is a fractional Brownian motion under the real world probability measure $\mathbf{P}$. Using the Riemann–Stieltjes Itô formula (2.1) and Wick–Itô–Skorohod Itô formula (3.3), to (4.2) and (4.3), respectively, we obtain the solutions

$$S_t = S_0 \exp\left(\int_0^t \mu(s)\,\mathrm{d}s + \sigma\tilde{Z}_t\right)$$

and

$$S_t = S_0 \exp\left(\int_0^t \left(\nu(s) - \sigma^2 H s^{2H-1}\right)\,\mathrm{d}s + \sigma\tilde{Z}_t\right).$$

Consequently, the dynamics (4.2) and (4.3) define the same stock price model if (and only if)

$$\mu(t) = \nu(t) - \sigma^2 H t^{2H-1}.$$

In the fractional Black–Scholes model there is no equivalent *martingale* measure. There is, however, a unique equivalent measure $\mathbf{Q}$ such that the solution to (4.2) or (4.3) is given by a *geometric fractional Brownian motion*

$$S_t := S_0 e^{Z_t - \frac{1}{2}t^{2H}}. \tag{4.4}$$

Here $\sigma = 1$ and Z is a $\mathbf{Q}$-fractional Brownian motion. The related Girsanov formula and the corresponding change of measure can be found in different forms e.g. in [3, 5, 8, 12]. It seems that the first formulation is due to Molchan and Golosov [11].
The definition (4.4) is analogous to (4.1) in the sense that

$$\mathbf{E}_{\mathbf{Q}} S_t = S_0$$

although S is not a martingale under $\mathbf{Q}$. For more information on this see [20, 21].
In the Riemann–Stieltjes sense the geometric fractional Brownian motion (4.4) is the solution to the equation

$$\mathrm{d}S_t = S_t(\,\mathrm{d}Z_t - Ht^{2H-1}\,\mathrm{d}t). \tag{4.5}$$

In the Wick–Itô–Skorohod sense the corresponding equation is

$$\delta S_t \;=\; S_t\,\delta Z_t. \tag{4.6}$$

To see that (4.4) is indeed the solution to (4.5) and (4.6) use the Itô formulas (2.1) and (3.3).
Henceforth, we will work under the "average risk neutral measure" $\mathbf{Q}$, i.e. we assume the dynamics (4.6), or equivalently (4.5), for the stock price.

5 Self-financing strategies and arbitrage

The Wick–Itô–Skorohod calculus is similar to the Itô calculus, e.g. integrals have always zero expectation. Thus, it has been proposed that one should use this calculus in the fractional pricing model (cf. Hu and Øksendal [8] and Bender [3]). In particular, it has been proposed that one should introduce the self-financing condition

$$\delta V_t(u) \;=\; u_t\,\delta S_t, \tag{5.1}$$

i.e.

$$V_t(u) \;=\; V_0(u) + \int_0^t u_s S_s\,\delta Z_s$$

for the wealth

$$V_t(u) \;:=\; u_t S_t + v_t$$

of a trading portfolio u. Here v is the (discounted) bank account. We will call this *pseudo self-financing* condition. Note that in (5.1) we use the interpretation

$$u_t\delta S_t \;=\; u_t S_t \delta Z_t.$$

To explain this, consider standard geometric Brownian motion. Let u be a predictable process such that the stochastic integral $\int_0^T u_t\,\mathrm{d}W_t$ exists as an Itô integral. It is well known that in this case $\int_0^T u_t\,\mathrm{d}W_t = \int_0^T u_t\,\delta W_t$. For the geometric Brownian motion S (4.1) we then have

$$\begin{aligned}\int_0^T u_t\,\mathrm{d}S_t &= \int_0^T u_t S_t\,\mathrm{d}W_t \\ &= \int_0^T u_t S_t\,\delta W_t \;=:\; \int_0^T u_t\,\delta S_t.\end{aligned}$$

Heuristically, the above interpretation means that the change of fortune on the interval $[t, t+\Delta]$ is $u_t S_t \diamond (W_{t+\Delta} - W_t)$, which in general is not the same as $u_t \diamond (S_{t+\Delta} - S_t)$.
One has no clear economic interpretation for the pseudo self-financing strategy. This is obvious from (3.2). But with this way of calculating the value of a self-financing portfolio

it was shown in [3, 8] that in the fractional Black–Scholes model one cannot generate arbitrage with self-financing strategies u satisfying the integrability condition

$$\int_0^T \left|\mathbf{E}\,(u_t S_t)^2\right|^{\frac{1}{H}}\,\mathrm{d}t \;<\; \infty.$$

Here H is the Hurst index of the fractional Brownian motion. If u is a pseudo self-financing trading portfolio in the Wick–Itô–Skorohod sense (5.1) we shall denote its wealth process by $V^{\mathsf{WIS}}(u)$. The freedom of arbitrage follows basically from the fact that the Wick–Itô–Skorohod integral is centered. So,

$$\mathbf{E}_{\mathbf{Q}} V_T^{\mathsf{WIS}}(u) \;=\; V_0^{\mathsf{WIS}}(u) \;=\; u_0 S_0 + v_0$$

and we cannot have riskless positive Wick–Itô–Skorohod wealth with zero initial capital. In the Riemann–Stieltjes sense the *self-financing* condition connected to the value of the portfolio u is of course

$$\mathrm{d}V_t(u) \;=\; u_t\,\mathrm{d}S_t, \tag{5.2}$$

i.e.

$$V_t(u) \;=\; V_0(u) + \int_0^t u_s S_s(\mathrm{d}Z_s - Hs^{2H-1}\,\mathrm{d}s).$$

Let us note that the condition (5.2) has a clear economical meaning in contrast to the pseudo self-financing condition. Indeed, one can consider the Riemann–Stieltjes integral as an almost sure limit of simple predictable trading strategies. When we use Riemann–Stieltjes self-financing condition to calculate the wealth associated to the portfolio u we write $V^{\mathsf{RS}}(u)$.

If u is a pseudo self-financing strategy that replicates a European claim f_T then

$$V_0(u) \;=\; \mathbf{E}_{\mathbf{Q}} V_T^{\mathsf{WIS}}(u) \;=\; \mathbf{E}_{\mathbf{Q}} f_T \tag{5.3}$$

since the Wick–Itô–Skorohod integral is centered. The relation (5.3) is not generally true for Riemann–Stieltjes self-financing portfolios as we shall see later in example *2*. Let us also note that in general we have

$$\begin{aligned} V_t^{\mathsf{RS}}(u) &\;\neq\; \mathbf{E}_{\mathbf{Q}}[f_T|\mathscr{F}_t], \\ V_t^{\mathsf{WIS}}(u) &\;\neq\; \mathbf{E}_{\mathbf{Q}}[f_T|\mathscr{F}_t] \end{aligned}$$

since neither of the value processes is a **Q**- martingale. For details how to compute $\mathbf{E}_{\mathbf{Q}}[f_T|\mathscr{F}_t]$ we refer to [20, 21].

Let us recall the connection between is the Riemann–Stieltjes and Wick–Itô–Skorohod integrals. For the proof of the following proposition we refer to Alòs et al. [2, Proposition 3 and the remark following it].

Proposition 5.1 *Let u be a stochastic process in $\mathbb{D}^{1,2}(|\mathscr{H}|)$ such that*

$$\int_0^T\int_0^T |D_s u_t|\;|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \;<\; \infty.$$

Assume also, that $u \in \mathscr{W}_q$ for some $q < 1/(1-H)$. Then the Riemann–Stieltjes integral exists and we have

$$\int_0^T u_t \,\mathrm{d}Z_t \;=\; \int_0^T u_t \,\delta Z_t + H(2H-1)\int_0^T\int_0^T D_s u_t \,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t. \tag{5.4}$$

So, the Riemann–Stieltjes integral is a Wick–Itô–Skorohod integral plus a "Malliavin trace".

Remark 5.2 Actually, Proposition 3 of [2] considers the so-called symmetric integral which is defined as a limit in probability. Nevertheless, in what follows we apply Proposition 5.1 only in cases when the Riemann–Stieltjes integral exists almost surely (and coincides with the symmetric integral).

Using (5.4) we obtain a formula for the difference of the value processes calculated under the two rather different self-financing conditions.

Theorem 5.3 *Let u be a trading portfolio such that uS satisfy the conditions of Proposition 5.1. Then*

$$V_T^{\mathsf{RS}}(u) - V_T^{\mathsf{WIS}}(u) \;=\; H(2H-1)\int_0^T S_t \int_0^T D_s u_t \,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t.$$

Proof: Recall that

$$V_T^{\mathsf{RS}}(u) \;=\; u_0 S_0 + v_0 + \int_0^T u_t S_t \,\mathrm{d}Z_t - H\int_0^T u_t S_t t^{2H-1}\,\mathrm{d}t. \tag{5.5}$$

By assumptions on uS we can use the Proposition 5.1. This yields

$$\begin{aligned} V_T^{\mathsf{WIS}}(u) \;&=\; u_0 S_0 + v_0 + \int_0^T u_t S_t \,\mathrm{d}Z_t \\ &\quad - H(2H-1)\int_0^T\int_0^T D_s(u_t S_t)\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{5.6}$$

Now, the Malliavin derivative D satisfies

$$D(FG) \;=\; F\,DG + G\,DF.$$

Moreover, $DS_t = S_t \mathbf{1}_{[0,t]}$. Thus,

$$D_s(u_t S_t) \;=\; u_t S_t \mathbf{1}_{[0,t]}(s) + S_t D_s u_t. \tag{5.7}$$

Inserting (5.7) to (5.6) and subsequently combining (5.5) and (5.6) we obtain

$$
\begin{aligned}
& V_T^{\mathsf{RS}}(u) - V_T^{\mathsf{WIS}}(u) \\
&= -H\int_0^T u_t S_t t^{2H-1}\,\mathrm{d}t \\
&\quad + H(2H-1)\int_0^T\int_0^T \left(u_t S_t \mathbf{1}_{[0,t]}(s) + S_t D_s u_t\right)|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&= -H\int_0^T u_t S_t t^{2H-1}\,\mathrm{d}t + H(2H-1)\int_0^T u_t S_t \int_0^t (t-s)^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + H(2H-1)\int_0^T S_t \int_0^T D_s u_t\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&= H(2H-1)\int_0^T S_t \int_0^T D_s u_t\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

This proves the theorem. □

Corollary 5.4 *Let u be Markovian trading strategy, i.e. $u_t = \gamma(t,S_t)$. Suppose that $\tilde{\gamma}$, defined as $\tilde{\gamma}(t,x) = \gamma(t, e^{x-\frac{1}{2}t^{2H}})$, satisfies the conditions of Proposition 3.1. Then*

$$
V_T^{\mathsf{RS}}(\gamma) - V_T^{\mathsf{WIS}}(\gamma) = H\int_0^T \frac{\partial\gamma}{\partial x}(t,S_t)S_t^2 t^{2H-1}\,\mathrm{d}t.
$$

Proof: The assumptions on γ allow us to use the Itô formula (3.3). However, we use Theorem 5.3 above. Let us calculate the Malliavin derivative of u. Because $DS_t = S_t\mathbf{1}_{[0,t]}$ we have

$$
D_s\gamma(t,S_t) = \frac{\partial\gamma}{\partial x}(t,S_t)S_t\mathbf{1}_{[0,t]}(s).
$$

So,

$$
\begin{aligned}
& H(2H-1)\int_0^T S_t \int_0^T D_s u_t\,|t-s|^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = H(2H-1)\int_0^T \frac{\partial\gamma}{\partial x}(t,S_t)S_t^2\int_0^t (t-s)^{2H-2}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = H\int_0^T \frac{\partial\gamma}{\partial x}(t,S_t)S_t^2 t^{2H-1}\,\mathrm{d}t.
\end{aligned}
$$

The claim follows. □

To illustrate the difference between the value of a portfolio using Riemann–Stieltjes value or the Wick–Itô–Skorohod value we calculate some examples.

1. Take a very simple portfolio, *buy-and-keep*, i.e. $u_t \equiv 1$. Of course

$$
S_0 + \int_0^T u_t\,\mathrm{d}S_t = S_T.
$$

This means that $V_T^{\mathsf{RS}}(u) = S_T$. Let us then calculate the Wick–Itô–Skorohod value of the portfolio u. Set

$$F(t,x) \;:=\; e^{x-\frac{1}{2}t^{2H}}.$$

Then, by using the Itô formula (3.3), we have that

$$\begin{aligned} S_0 + \int_0^T u_t\,\delta S_t &= S_0 + \int_0^T S_t\,\delta Z_t \\ &= S_0 + \int_0^T F(t,Z_t)\,\delta Z_t \\ &= S_0 + \int_0^T \frac{\partial F}{\partial x}(t,Z_t)\,\delta Z_t \\ &= F(T,Z_T) - \int_0^T \left(\frac{\partial F}{\partial t}(t,Z_t) + Ht^{2H-1}\frac{\partial^2 F}{\partial x^2}(t,Z_t)\right)\mathrm{d}t \\ &= S_T. \end{aligned}$$

In the calculations above we used the fact that F satisfi es the equations

$$F(t,x) = \frac{\partial F}{\partial x}(t,x) \quad \text{and} \quad \frac{\partial F}{\partial t}(t,x) = -Ht^{2H-1}F(t,x).$$

The result $V_T^{\mathsf{RS}}(u) = V_T^{\mathsf{WIS}}(u)$ remains true for any *deterministic* u. Actually, the above computation was unnecessary. Indeed, $Du \equiv 0$ for any deterministic u. Consequently, the the Riemann–Stieltjes and Wick–Itô–Skorohod values coincide by Theorem 5.3.

2. Let us consider now a non-deterministic portfolio. Take $u_t = S_t - S_0$. Note that we start with no capital. Then by the Riemann–Stieltjes change of variables formula (2.1) we have

$$\begin{aligned} V_T^{\mathsf{RS}}(u) &= \int_0^T (S_t - S_0)\,\mathrm{d}S_t \\ &= \frac{1}{2}(S_T - S_0)^2. \end{aligned}$$

However, we have $u_t = \gamma(S_t)$ with $\gamma(x) = x - S_0$. So, Corollary 5.4 yields

$$V_T^{\mathsf{WIS}}(u) \;=\; V_T^{\mathsf{RS}}(u) - H\int_0^T S_t^2 t^{2H-1}\,\mathrm{d}t.$$

Hence, we obtain that $V_T^{\mathsf{WIS}} < V_T^{\mathsf{RS}}$. Note that this strategy u is the arbitrage example of Dasgupta and Kallianpur [4] and Shiryaev [17]. For more information on arbitrage in the Riemann–Stieltjes model we refer to Norvaiša [13] and Salopek [16].
Let us consider Δ-hedging of a Markovian claim $f_T = f_T(S_T)$. By the Itô formula (3.3) we have

$$V_T^{\mathsf{WIS}}\left(\frac{\partial\gamma}{\partial x}\right) \;=\; \gamma(T,S_T) - \int_0^T \left(\frac{\partial \gamma}{\partial t}(t,S_t) + HS_t^2 t^{2H-1}\frac{\partial^2\gamma}{\partial x^2}(t,S_t)\right)\mathrm{d}t. \qquad (5.8)$$

So, if γ satisfies the *fractional Black–Scholes differential equation*

$$\frac{\partial \gamma}{\partial t}(t,x) \;=\; -Hx^2 t^{2H-1}\frac{\partial^2\gamma}{\partial x^2}(t,x) \tag{5.9}$$

with the boundary condition $\gamma(T,x) = f_T(x)$ then $\frac{\partial\gamma}{\partial x}$ replicates the claim f_T. Moreover, $\gamma(0,S_0)$ is the corresponding fair price. Note however that by Corollary 5.4 this Skorohod replication is actually a super replication in the Riemann–Stieltjes sense if the claim f_T is convex.

3. Consider Δ-hedging on a European call option $(S_T-K)^+$. Motivated by the classical Black–Scholes model Hu and Øksendal [8] introduced the function

$$h(t,x) = x\,\Phi\left(\frac{\log\frac{x}{K}+\frac{1}{2}\left(T^{2H}-t^{2H}\right)}{\sqrt{T^{2H}-t^{2H}}}\right) - K\Phi\left(\frac{\log\frac{x}{K}-\frac{1}{2}\left(T^{2H}-t^{2H}\right)}{\sqrt{T^{2H}-t^{2H}}}\right) \tag{5.10}$$

where Φ is the cumulative probability distribution function of a standard normal law, i.e.

$$\Phi(z) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,\mathrm{d}y.$$

Now h satisfies the fractional Black–Scholes differential equation (5.9) Therefore, the $\mathrm{d}t$-integral in (5.8) vanishes and in the Wick–Itô–Skorohod sense we have

$$V_T^{\mathsf{WIS}}\left(\frac{\partial h}{\partial x}\right) \;=\; h(T,S_T).$$

Since $h(T,S_T) = (S_T-K)^+$ we have found a Wick–Itô–Skorohod replicating portfolio for the European call option and the corresponding fair price

$$C_T \;=\; h(0,S_0).$$

However, Corollary 5.4 tells us that

$$V_T^{\mathsf{RS}}\left(\frac{\partial h}{\partial x}\right) \;=\; (S_T-K)^+ + \int_0^T \frac{\partial^2 h}{\partial x^2}(t,S_t)S_t^2 t^{2H-1}\,\mathrm{d}t.$$

Since $\frac{\partial^2 h}{\partial x^2}$ is positive we see that the portfolio $\frac{\partial h}{\partial x}$ is actually a super replication for $(S_T-K)^+$ in the Riemann–Stieltjes sense.

4. Let us consider the Δ-hedging of the European put option $(K-S_T)^+$. Recall that the Wick–Itô–Skorohod integral is centered. Thus, for the call option we have

$$C_T \;=\; \mathbf{E}_{\mathbf{Q}}(S_T-K)^+.$$

Consequently, the equality

$$(K-S_T)^+ \;=\; (S_T-K)^+ \,+\, (K-S_T)$$

yields the *call–put parity*

$$P_T \;=\; C_T \,+\, (K - S_0) \tag{5.11}$$

for the fair price P_T of the European put option. The corresponding replicating portfolio can be constructed then by using the function h introduced in (5.10). Indeed,

$$\begin{aligned}(K - S_T)^+ &= (S_T - K)^+ \,+\, (K - S_T) \\ &= h(T, S_T) + K - S_T \\ &= h(0, S_0) + \int_0^T \frac{\partial h}{\partial x}(t, S_t)\,\delta S_t + K - \left(S_0 + \int_0^T 1\,\delta S_t\right) \\ &= h(0, S_0) + K - S_0 + \int_0^T \left(\frac{\partial h}{\partial x}(t, S_t) - 1\right)\delta S_t.\end{aligned}$$

So, the Δ-hedge is $\frac{\partial \tilde{h}}{\partial x}$, where

$$\tilde{h}(t, x) \;=\; h(t, x) + K - x.$$

The Riemann–Stieltjes value of the portfolio $\frac{\partial \tilde{h}}{\partial x}$ is of course obtained by Corollary 5.4, viz.

$$\begin{aligned}V_T^{\mathsf{RS}}\left(\frac{\partial \tilde{h}}{\partial x}\right) &= (K - S_T)^+ + \int_0^T \frac{\partial^2 \tilde{h}}{\partial x^2}(t, S_t) S_t^2 t^{2H-1}\,\mathrm{d}t \\ &= (K - S_T)^+ + \int_0^T \frac{\partial^2 h}{\partial x^2}(t, S_t) S_t^2 t^{2H-1}\,\mathrm{d}t.\end{aligned}$$

So, in the Riemann–Stieltjes sense the strategy $\tilde{h}$ is a super replication for the European put generating the same amount of extra money as the corresponding hedge in the call option case.

6 Conclusions

If one is happy with the Wick–Itô–Skorohod definition of a self-financing portfolio then the fractional Black–Scholes model is free of arbitrage. Let us note, however, that the proof of the freedom of arbitrage in [3, 8] does not assume that the portfolio is adapted to the filtration generated by the stock price process. So, in principle one cannot generate arbitrage even though one knows the future values of the stock. Also, it should be noted that e.g. the arbitrage opportunity constructed by Rogers [15] does not depend on any particular notion of integration. The same is true for the pre-limit arbitrage of fractional Black–Scholes model considered in [19].

On the other hand, under the Riemann–Stieltjes notion of self-financing there is arbitrage in the fractional Black–Scholes model. So, the questions of fair price of an option or replication become unclear, even meaningless. Indeed, suppose that there is a minimal hedge for you favourite European option. Then combining that hedge with an arbitrage strategy we obtain a super replication with the same initial capital as the (supposed) minimal hedge has. Thus, it is not reasonable to call the initial wealth of the replicating portfolio the fair price of the option.

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