Representations of Gaussian bridges

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Contents

Brownian bridge

Let $o = (0, \xi) \to (T, \theta)$

$$
\begin{aligned}
\mathrm{d}W_t^o &= \mathrm{d}W_t + \frac{\theta - W_t^o}{T-t}\mathrm{d}t, \quad W_0^o = \xi, \qquad (sde) \\
W_t^o &= \xi + (\theta - \xi)\frac{t}{T} + (T-t)\int_0^t \frac{\mathrm{d}W_s}{T-s}, \\
W_t^o &= \theta\frac{t}{T} + \left(W_t - \frac{t}{T}W_T\right).
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathrm{Law}(W^o; \mathbf{P}) &= \mathrm{Law}(W; \mathbf{P}^o), \\
\mathbf{P}^o &= \mathbf{P}(\cdot | W_T = \theta).
\end{aligned}
$$

Setting $\theta = W_T^o$ we see that (sde) is the semimartingale decomposition of W^o in the the filtration $\mathcal{F}_t^{W^o} \vee \sigma\{W_T^o\}$ and W is a Brownian motion in this filtration.

General anticipative representation

Let X be Gaussian with mean μ and covariance R. Then $X^o = X | X_T = \theta$ is Gaussian with with

$$\mathbf{E}(X_t | X_T = \theta) =$$

$$\frac{R(T,t)}{R(T,t)} (\theta - \mu(T)) + \mu(t),$$

$$\mathbf{Cov}(X_t, X_s | X_T = \theta) =$$

$$R(t,s) - \frac{R(T,t)R(T,s)}{R(T,T)}.$$

From the orthogonal decomposition of X given X_T we obtain an anticipative representation for any Gaussian bridge:

$$X_t^o = \theta \frac{R(T,t)}{R(T,T)} + \left(X_t - \frac{R(T,t)}{R(T,T)} X_T \right).$$

Abstract non-anticipative representation

Idea is to use the prediction martingale m of X and the Girsanov's theorem.

Assumptions:

(A0) Filtration of X is continuous

(A1) $\mathbf{P}_t^o \sim \mathbf{P}_t$ for all $t < T$.

(A2) The non-anticipative linear functional

$$F_T : X_t \mapsto m_t = \mathbf{E}(X_T | \mathcal{F}_t^X)$$

is injective

Abstract non-anticipative representation, cont.

Denote $\langle m\rangle_{T,t} := \langle m\rangle_T - \langle m\rangle_t$.

By using the Bayes' rule and Itô's formula we see that

$$\mathrm{d}\mathbf{P}_t^o = L_t^o \mathrm{d}\mathbf{P}_t,$$

where

$$\log L_t =$$

$$\int_0^t \frac{\theta - m_s}{\langle m\rangle_{T,s}} \mathrm{d}m_s - \frac{1}{2}\int_0^t \left(\frac{\theta - m_s}{\langle m\rangle_{T,s}}\right)^2 \mathrm{d}\langle m\rangle_s.$$

Let $S_o(m)$ be the solution of

$$\mathrm{d}m_t^o = \mathrm{d}m_t + \frac{\theta - m_t^o}{\langle m\rangle_{T,t}} \mathrm{d}\langle m\rangle_t, \quad m_0^o = \zeta,$$

i.e.

$$m_t = S_o(m)_t =$$

$$\zeta + (\theta - \zeta)\frac{\langle m\rangle_t}{\langle m\rangle_T} + \langle m\rangle_{T,t}\int_0^t \frac{\mathrm{d}m_s}{\langle m\rangle_{T,s}}.$$

Abstract non-anticipative representation, cont., cont.

By using the Girsanov's theorem we see that if X satisfies (A0), (A1) and (A2) then

$$X^o = F_T^{-1} \circ S_o \circ F_T(X).$$

This representation is non-anticipative.

Bridges of Gaussian martingales

Let M be a continuous Gaussian martingale with strictly increasing bracket $\langle M\rangle$ and $M_0 = \xi$.

$$\begin{aligned}
\mathrm{d}M_t^o &= \mathrm{d}M_t + \frac{\theta - M_t^o}{\langle M\rangle_{T,t}}\mathrm{d}\langle M\rangle_t, \quad M_0^o = \xi, \\
M_t^o &= \xi + (\theta - \xi)\frac{\langle M\rangle_t}{\langle M\rangle_T} + \langle M\rangle_{T,t}\int_0^t \frac{\mathrm{d}M_s}{\langle M\rangle_{T,s}}, \\
M_t^o &= \theta\frac{\langle M\rangle_t}{\langle M\rangle_T} + \left(M_t - \frac{\langle M\rangle_t}{\langle M\rangle_T}M_T\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathbf{E}M_t^o &= \xi + (\theta - \xi)\frac{\langle M\rangle_t}{\langle M\rangle_T}, \\
\mathbf{Cov}(M_t^o, M_s^o) &= \langle M\rangle_{t\wedge s} - \frac{\langle M\rangle_t\langle M\rangle_s}{\langle M\rangle_T}.
\end{aligned}$$

To see this just note that $R(t,s) = \langle M\rangle_{t\wedge s}$ and F_T is, of course, the identity.

Bridges of Wiener predictable processes

Assume:

$$\text{(A3)} \qquad m_t = \int_0^t p_T(t,s)\,\mathrm{d}X_s,$$

$$\text{(A4)} \qquad X_t = \int_0^t p_T^*(t,s)\,\mathrm{d}m_s.$$

Given (A0) [cts. filtration], (A1) [$\mathbf{P}_t^o \sim \mathbf{P}_t$], (A3) and (A4):

$$X_t^o = X_t + \int_0^t \left\{\theta - \int_0^s p_T(s,u)\,\mathrm{d}X_u^o\right\} \frac{p_T^*(t,s)}{\langle m\rangle_{T,s}}\,\mathrm{d}\langle m\rangle_s.$$

$$X_t^o = \theta\frac{R(T,t)}{R(T,T)} + X_t - \int_0^t \phi_T(t,s)\,\mathrm{d}X_s,$$

$$\phi_T(t,s) =$$

$$\int_s^t \left\{\int_s^u \frac{p_T(v,s)}{\langle m\rangle_{T,v}^2}\,\mathrm{d}\langle m\rangle_v - \frac{p_T(u,s)}{\langle m\rangle_{T,u}}\right\} p_T^*(t,u)\,\mathrm{d}\langle m\rangle_u.$$

Bridges of Volterra processes

(A5) There exists a Volterra kernel k and a continuous Gaussian martingale M with strictly increasing bracket $\langle M \rangle$ such that

$$X_t = \int_0^t k(t,s)\mathsf{d}M_s.$$

Let K extend $\mathbf{1}_{[0,t)} \mapsto k(t,\cdot)$ linearly and assume:

(A6) The equation $\mathsf{K}f = \mathbf{1}_{[0,t)}$ has a solution.
(A7) The equation $\mathsf{K}g = \mathbf{1}_{[0,t)}k(T,\cdot)$ has a solution.

By (A6) we may set $k^*(t,s) = \mathsf{K}^{-1}\mathbf{1}_{[0,t)}(s)$ we have

$$M_t = \int_0^t k^*(t,s)\mathsf{d}X_s.$$

Since

$$\mathsf{d}m_t = k(T,t)\mathsf{d}M_t,$$

we have

$$X_t = \int_0^t \frac{k(t,s)}{k(T,s)}\mathsf{d}m_s$$

Bridges of Volterra processes, cont.

By (A7) we have

$$m_t = X_t + \int_0^t \Psi_T(t,s)\mathsf{d}X_s,$$

$$\Psi_T(t,s) = \mathsf{K}^{-1}\left[\mathbf{1}_{[0,t)}k(T,\cdot)\right](s).$$

So, we have found that

$$\begin{aligned} \mathsf{d}\langle m\rangle_t &= k(T,t)^2\mathsf{d}\langle M\rangle_t, \\ p_T(t,s) &= \mathbf{1}_{[0,t)}(s) + \Psi_T(t,s), \\ p_T^*(t,s) &= \frac{k(t,s)}{k(T,s)}. \end{aligned}$$

These functions are known explicitly if X is, for example, the fractional Brownian motion or the Riemann–Liouville process