Probability is irrelevant in stochastic finance

Black-Scholes model is correct despite of stylized facts

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Outline

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Let $(B_t)_{t \in [0,T]}$ be the bond, or BANK ACCOUNT. We work in the discounted world:

$$B_t = 1 \qquad \text{for all } t \in [0, T].$$

The bank account is RISKLESS, i.e. non-random.

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Let $S = (S_t)_{t\in[0,T]}$ be the STOCK. The stock is RISKY, i.e. random.

We assume that S

1. is positive and continuous in time,
2. has non-trivial QUADRATIC VARIATION of form

$$(\mathrm{d}S_t)^2 \ = \ \sigma^2 S_t^2 \,\mathrm{d}t.$$

Definition (Option)

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Example

- $G = (S_T - K)^+$ is a call-option,
- $G = (K - S_T)^+$ is a put-option,
- $G = S_T - K$ is a future.

T is the time of maturity and K is the strike-price.

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where the differentials are of "forward type". Indeed, let d be the infinitesimal. Then the differential equation above can be written as

$$V_{t+\mathrm{d}} \;=\; \Phi_t S_{t+\mathrm{d}} + (V_t - \Phi_t S_t)\,.$$

Replication principle is used to hedge and price options.

Definition (Replication principle)

Let G be an option. Suppose that there is a trading strategy Φ with wealth $V_t(\Phi)$ at time t such that $G = V_T(\Phi)$ at time T. Then the price of the option G at time t is $V_t(\Phi)$.

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The replication requirement $G = V_T(\Phi)$ can be written as

$$G = V_t(\Phi) + \int_t^T \Phi_s \, \mathrm{d}S_s,$$

where the integral is of "forward type".

Theorem (Ito's Formula)

The following is true when at least three terms in the equation make sense in the classical way:

$$
\begin{aligned}
\mathrm{d}f(t, S_t) &= \frac{\partial f}{\partial t}(t, S_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, S_t)\mathrm{d}S_t \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)(\mathrm{d}S_t)^2.
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Proof.

Taylor is all you need! □

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Recall that

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So, by Ito's formula, if $v(t,x)$ is the solution to Black-Scholes BPDE

$$\frac{\partial v}{\partial t}(t,x) - \frac{\sigma^2}{2} x^2 \frac{\partial^2 v}{\partial x^2}(t,x) \;=\; 0$$

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with boundary condition $v(T, x) = g(x)$, then

$$V_t(\Phi) \;=\; v(t, S_t)$$

and

$$\Phi_t \;=\; \frac{\partial v}{\partial x}(t, S_t).$$

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Note that the prices and replications of (path-independent) options were derived from the quadratic variation property. Probability does not come into it!

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$$v(t,x) \;=\; \mathbf{E}\left[g\left(x e^{\sigma W_{T-t} - \frac{\sigma^2}{2}(T-t)} \right) \right],$$

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but beware the CLT Non Sequitur:

> *It is Gaussian. Therefore, it is a sum of small independent parts.*

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In practise the log-returns are neither normally distributed nor independent. (Believe it! Everybody wants them to be, so there have been a lot of arguments why they should be. Unfortunately, the objective reality, i.e. the data, disagrees with theory.)

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But now we know that the Black-Scholes prices and replications may indeed be right even though the model is obviously wrong.

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In the Black-Scholes model the volatility is the variance of the log-returns, so one usually estimates it from the data by using the HISTORICAL VOLATILITY i.e. one takes the empirical variance as the estimate.

USING THE HISTORICAL VOLATILITY IS WRONG! Indeed, consider the model

$$\mathrm{d}S_t = S_t \sigma \mathrm{d}(W_t + Z_t)$$

where Z is a long-range-dependent fractional Brownian motion independent of the standard Brownian motion. Then the volatility of this models is σ^2, but the variance of the log-returns is $2\sigma^2$.

- The End -