What Is Volatility?

Probability Is Irrelevant in Option-Pricing

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To illustrate the problem we construct a toy-model that incorporates LONG-RANGE DEPENDENCE and HEAVY TAILS to the standard Black–Scholes model while keeping the replication prices of options unchanged.

So, the volatility as the pricing parameter is the same as in the classical Black–Scholes model, but the historical volatility (standard deviation) is not the same as in the Black–Scholes model. Indeed, the historical volatility may not even exist.

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The talk is based on C. Bender, T. Sottinen, and E. Valkeila (2008): Pricing by hedging and no-arbitrage beyond semimartingales. *Finance and Stochastics*, forthcoming.

Outline

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We consider discounted markets with one risky asset given by the mixed model

$$S_t = s_0 \exp\left\{\mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B_t^H - I_t^{\alpha_1} + I_t^{\alpha_2}\right\},$$

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- B^H is a FRACTIONAL BROWNIAN MOTION with Hurst index $H > 0.5$.
- I^{α_i}'s are INTEGRATED COMPOUND POISSON PROCESSES with positive heavy-tailed jumps:

$$I_t^{\alpha_i} = \int_0^t \sum_{k:\tau_k^i \le s} U_k^i \, \mathrm{d}s,$$

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  τ_k^i's are Poisson arrivals and $\mathbf{P}[U_k^i > x] \sim x^{-\alpha_i}$.
- W, B^H, I^{α_1}, and I^{α_2} are independent.

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2. HEAVY TAILS: $\mathbf{P}[-R_t > x] \gtrsim x^{-\alpha_1}$ and $\mathbf{P}[R_t > x] \gtrsim x^{-\alpha_2}$.
3. GAIN/LOSS ASYMMETRY: Obvious if $\alpha_1 < \alpha_2$.
4. JUMPS: No, but can you tell the difference between jumps and heavy tails from a discrete data?
5. VOLATILITY CLUSTERING: What is volatility? If volatility is STANDARD DEVIATION, we can get any kind of volatility structure: Change the Poisson arrivals to clustered arrivals. If volatility (squared) is the so-called QUADRATIC VARIATION then it is fixed to constant σ^2.

Outline

The forward integral is economically meaningful in the context of self-financing strategies:

Let (π_n) is a fixed sequence of, say, dyadic partitions of $[0, T]$. Then the Forward Integral

$$\int_0^t \Phi_u \, \mathrm{d}S_u$$

(along the sequence of partitions (π_n)) is the **P**-a.s. forward-sum limit

$$\lim_{n \to \infty} \sum_{\substack{t_k \in \pi_n \\ t_k \leq t}} \Phi_{t_{k-1}} \left(S_{t_k} - S_{t_{k-1}} \right)$$

(when it exists).

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$$\lim_{n \to \infty} \sum_{\substack{t_k \in \pi_n \\ t_k \leq t}} \left(S_{t_k} - S_{t_{k-1}} \right)^2$$

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4. $$\langle g \circ X \rangle_t = \int_0^t g'(X_u)\mathrm{d}\langle X \rangle_u \,.$$

Theorem

Let $f \in \mathcal{C}^{1,2}([0,T], \mathbb{R}_+)$. If S has quadratic variation then we have the Itô formula

$$\mathrm{d}f(t, S_t) = f_t(t, S)\mathrm{d}t + f_x(t, S)\mathrm{d}S_t + \frac{1}{2} f_{xx}(t, S_t)\mathrm{d}\left\langle S \right\rangle_t$$

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Proof.

Taylor is all you need. $\square$

Remark

Itô's formula implies that the forward integral on the right hand side exists and has a continuous modification.

Theorem

Let $F(S_T)$ be a European option with maturity T. Let $f(t, S_t)$ satisfy the Black-Scholes BPDE

$$f_t(t, x) + \frac{\sigma^2 x^2}{2} f_{xx}(t, x) = 0, \quad f(T, x) = F(x).$$

Then $f_x(t, S_t)$ is the Delta-hedge for $F(S_T)$ and $f(0, s_0)$ is the price of the option.

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Proof.

Note that $\mathrm{d}\langle S\rangle_t = \sigma^2 S_t^2 \mathrm{d}t$, and then Itô is all you need. □

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2. The FEYNMAN-KAC connection to BPDEs tells us that

$$f(t,x) = \mathbf{E}\left[F(\tilde{S}_T) \,\middle|\, \tilde{S}_t = x\right],$$

   where $\tilde{S}$ is the Geometric Brownian Motion. This is true despite of the facts that our toy model is not log-normal, and the returns are not independent.

Outline

Let R_k's be the log-returns

$$R_k = \log \frac{S_k}{S_{k-1}}, \qquad k = 1, 2, \ldots .$$

- Then
$$\mathbf{Var}[R_k] = \sigma^2 + \delta^2 + v_{k1}^2 + v_{k2}^2,$$
where $v_{k1}^2, v_{k2}^2 \to \infty$ (possibly already $+\infty$ for finite k) are the variances of the increments of I^{α_1} and I^{α_2}.

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where $v_{k1}^2, v_{k2}^2 \to \infty$ (possibly already $+\infty$ for finite k) are the variances of the increments of I^{α_1} and I^{α_2}.
- We have that
$$\frac{1}{n}\sum_{k=1}^{n}\left(R_k - \frac{1}{n}\sum_{k=1}^{n} R_k\right)^2 \to \sigma^2 + \delta^2 > \sigma^2$$
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- So, HISTORICAL VOLATILITY $\neq$ QUADRATIC VARIATION.

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- The IMPLIED VOLATILITY IS INDEPENDENT OF THE "SMOOTH" PARTS B^H, I^{α_1}, AND I^{α_2}.
- The quadratic variation is independent of probabilistic properties.
- Probability is irrelevant in option pricing and replication.

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- A naïve approach to estimate the implied volatility historically would then be to use

$$\hat{\sigma}_n^2 = \frac{1}{2^n} \sum_{k=1}^{2^n} \left(R_{kT/2^n} - \frac{1}{2^n} \sum_{k=1}^{2^n} R_{kT/2^n} \right)^2 .$$

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- So, in estimating variance one lets time go to infinity, while in estimating quadratic variation one lets time-increments go to zero.
- With financial time series the naïve historical volatility estimation does not work: There is no price process in the microscopic level!

Outline

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