Are stylized facts irrelevant in option-pricing?

Kyiv, June 19-23, 2006

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Based on a joint work

No-arbitrage pricing beyond semimartingales

with

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Outline

1. Market models, and self-financing strategies

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2. Pricing with replication, and arbitrage

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1. Market models, and self-financing strategies

- Let $\mathcal{C}_{s_0,+}$ be the space of continuous positive paths $\eta : [0, T] \to \mathbb{R}$ with $\eta(0) = s_0$.
  A discounted market model is five-tuple $(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbf{P})$ where the stock-price process S takes values in $\mathcal{C}_{s_0,+}$.

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  A discounted market model is five-tuple $(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbf{P})$ where the stock-price process S takes values in $\mathcal{C}_{s_0,+}$.
- Non-anticipating trading strategy Φ is self-financing if its wealth satisfies

$$V_t(\Phi, v_0; S) = v_0 + \int_0^t \Phi_r \, \mathrm{d}S_r, \quad t \in [0, T]. \tag{1}$$

  Here the economic notion 'self-financing' is captured by the 'forward' construction of the pathwise integral in (1).

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- A strategy Φ is arbitrage (free lunch) if

$$\mathbf{P}\left[V_T(\Phi, 0; S) \geq 0\right] = 1 \quad \text{and} \quad \mathbf{P}\left[V_T(\Phi, 0; S) > 0\right] > 0.$$

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- If the hedging capital v_0 is not unique them there is strong arbitrage. Also, note that replication and arbitrage are kind of opposite notions.

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All of these stylized facts are in conflict with the Black–Scholes model, and they are ill suited for semimartingale models.

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1. S takes values in $\mathcal{C}_{s_0,+}$,
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$$\mathrm{d}\langle S \rangle_t \;=\; \sigma^2 S_t^2 \,\mathrm{d}t,$$

3. for all $\varepsilon > 0$ and $\eta \in \mathcal{C}_{s_0,+}$ we have the small ball property

$$\mathbf{P}\left[\|S - \eta\|_\infty < \varepsilon\right] > 0.$$

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$$\lim_{n\to\infty} \sum_{\substack{t_k\in\pi_n \\ t_k\le t}} \Phi_{t_{k-1}} \left(S_{t_k} - S_{t_{k-1}} \right).$$

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- Let $u \in \mathcal{C}^{1,2,1}([0,T], \mathbb{R}_+, \mathbb{R}^m)$ and $Y^1, \ldots, Y^m$ be bounded variation processes. If S has pathwise quadratic variation then we have the Itô formula for $u(t, S_t, Y_t^1, \ldots, Y_t^m)$:

$$\mathrm{d}u \;=\; \frac{\partial u}{\partial t}\,\mathrm{d}t + \frac{\partial u}{\partial x}\,\mathrm{d}S + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}\langle S\rangle + \sum_{i=1}^m \frac{\partial u}{\partial y_i}\,\mathrm{d}Y^i.$$

  This implies that the forward integral on the right hand side exists and has a continuous modification.

7. Allowed strategies

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A strategy Φ is allowed if it is admissible and of the form

$$\Phi_t \;=\; \varphi\left(t, S_t, g_1(t,S), \ldots, g_m(t,S)\right),$$

where $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}_+ \times \mathbb{R}^m)$ and g_k's are hindsight factors:

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1. $g(t,\eta) = g(t,\tilde{\eta})$ whenever $\eta(r) = \tilde{\eta}(r)$ on $r \in [0,t]$,
2. $g(\cdot,\eta)$ is of bounded variation and continuous,
3. $\left|\int_0^t f(u)\mathrm{d}g(u,\eta) - \int_0^t f(u)\mathrm{d}g(u,\tilde{\eta})\right| \leq K\|f\mathbf{1}_{[0,t]}\|_\infty \|\eta - \tilde{\eta}\|_\infty$

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8. A no-arbitrage and robust-hedging result

Theorem NA *There is no arbitrage with allowed strategies.*

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Theorem RH *Suppose a continuous option $G : \mathcal{C}_{s_0,+} \to \mathbb{R}$. If $G(\tilde{S})$ can be hedged in one model $\tilde{S} \in \mathcal{M}_\sigma$ with an allowed strategy then $G(S)$ can be hedged in any model $S \in \mathcal{M}_\sigma$.*

Moreover, the hedges are – as strategies of the stock-path – independent of the model.

Moreover still, if φ is a 'functional hedge' in one model then it is a 'functional hedge' in all models.

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Corollary PDE *In the Black–Scholes model hedges for European, Asian, and lookback-options can be constructed by using the Black–Scholes partial differential equation. These hedges hold for any model that is continuous, satisfies the small ball property, and has the same quadratic variation as the Black–Scholes model.*

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Consider a mixed model

$$S_t = s_0 \exp\left\{\mu t + \sigma W_t - \frac{\sigma}{2} t + \delta B_t^H - I_t^{\alpha_1} + I_t^{\alpha_2}\right\},$$

where

- B^H is a fractional Brownian motion with Hurst index $H > 0.5$.

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- B^H is a fractional Brownian motion with Hurst index $H > 0.5$.
- I^{α_i}'s are integrated compound Poisson processes with positive heavy-tailed jumps:

$$I_t^{\alpha_i} = \int_0^t \sum_{k:\tau_k^i \le s} U_k^i \, \mathrm{d}s,$$

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  τ_k^i's are Poisson arrivals and $\mathbf{P}[U_k^i > x] \sim x^{-\alpha_i}$.
- W, B^H, I^{α_1}, and I^{α_2} are independent.

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9. Mixed models with stylized facts (2/2)

Consider now stylized facts in the mixed model.

1. Long-range dependence: If I^{α_i}'s are in L^2 then

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4. Jumps: No, but can you tell the difference between jumps and heavy tails from a discrete data?
5. Volatility clustering: What is volatility? If volatility is standard deviation, we can have any kind of volatility structure: E.g. change the Poisson arrivals to clustered arrivals. If volatility (squared) is the quadratic variation then it is fixed to constant σ^2.

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- The quadratic variation is a path property. It tells nothing about the probabilistic structure of the stock-price (Black and Scholes tell us the mean return is irrelevant. We boldly suggest that probability is irrelevant, as far as option-pricing is concerned).
- Don't be surprised if the implied and historical volatility do not agree: The latter is an estimate of the variance and the former is an estimate of the quadratic variation. In the Black–Scholes model these notions coincide. But that is just luck! Indeed, consider a mixed fractional Black–Scholes model $R_t = \sigma \Delta W_t + \delta \Delta B_t^H$. Then quadratic variation or R_t is σ^2, but the variance of R_t is $\sigma^2 + \delta^2$.

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- Don't use the historical volatility! Instead, use either implied volatility or estimate the quadratic variation (which may be difficult).

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- Instead of taking the Black–Scholes model as reference we can consider models

$$\tilde{S}_t = s_0 \exp \tilde{X}_t,$$

where $\tilde{X}$ is continuous semimartingale with $\tilde{X}_0 = 0$.

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where $\tilde{X}$ is continuous semimartingale with $\tilde{X}_0 = 0$.
- We can extend our robustness results to models
$$S_t = s_0 \exp X_t$$
where X is continuous $X_0 = 0$, X and $\tilde{X}$ have the same pathwise quadratic variation, and the support of $\mathbf{P} \circ X^{-1}$ is the same as the support of $\tilde{\mathbf{P}} \circ \tilde{X}^{-1}$.

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where X is continuous $X_0 = 0$, X and $\tilde{X}$ have the same pathwise quadratic variation, and the support of $\mathbf{P} \circ X^{-1}$ is the same as the support of $\tilde{\mathbf{P}} \circ \tilde{X}^{-1}$.
- So, when option pricing is considered it does not matter whether $\tilde{S}$ or S is the model.

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