Replication and Absence of Arbitrage in Non-Semimartingale Models

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Outline

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4. Mixed fractional Brownian motion.
5. Replication and arbitrage in non-semimartingale models. [Joint work with C. Bender, WIAS, Berlin;and E. Valkeila, Helsinki University of Technology.]
   This is the part with new results.

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1. The classical pricing model: Black & Scholes model.
2. Geometric fractional Brownian motion as a pricing model.
3. Why not geometric fractional Brownian motion?
4. Mixed fractional Brownian motion.
5. Replication and arbitrage in non-semimartingale models. [Joint work with C. Bender, WIAS, Berlin;and E. Valkeila, Helsinki University of Technology.]
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6. Closing remarks.

The basic pricing model: Black & Scholes model

- The *riskless bond* has dynamics

$$dB_t = rB_t dt,\ \ B_0 = 1;$$

and the *risky stock* has dynamics

$$dS_t = S_t(\mu dt + \sigma dW_t),\ \ S_0 = s > 0.$$

Here r is the short rate, σ is the volatility parameter, μ is the growth rate, and W is a *Brownian motion*.

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  Here r is the short rate, σ is the volatility parameter, μ is the growth rate, and W is a *Brownian motion*.
- The option $f(S_T)$ has price v^f

  $$v^f = e^{-rT}\mathbf{E}_{\mathbf{Q}}[f(S_T)];$$

  here $\mathbf{Q}$ is the *equivalent martingale measure*.

The basic pricing model: Black & Scholes model

- The *replication price* is the same as the risk neutral price v_f:

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f(S_T) &= V_T(\Phi, v^f; S) \\
&= v^f + \int_0^T \Psi_s dB_s + \int_0^T \Phi_s dS_s.
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- The *self-financing hedge* Φ is obtained from

$$C(t,x) = e^{-r(T-t)} \mathbf{E}_{\mathbf{Q}}[f(S_T) | F_t^S]_{S_t = x}$$

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- Properties of the Black & Scholes model:
  – log-returns are independent.
  – log-returns are Gaussian.

Geometric fractional Brownian motion

- To model dependence of the log-returns we could use fractional Brownian motion B^H. It is a continuous Gaussian process with mean zero and covariance

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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- Brownian motion W is a special case of fractional Brownian motion B^H with the parameter value $H = \frac{1}{2}$.

Geometric fractional Brownian motion

- Fractional Brownian motion is not a semimartingale, but one can show for $H > \frac{1}{2}$ the stochastic differential equation

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has the solution

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- For $H > \frac{1}{2}$ the increments of B^H are positively correlated.

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- The *fundamental theorem of asset pricing* states that "no-arbitrage" means "existence of an equivalent martingale measure." So, non-semimartingales are ruled out as models for stock.
- Fractional Brownian motion is not a semimartingale. Therefore, the geometric fractional Browanian motion is not a semimartingale.
- Explicit arbitrage examples with geometric fractional Brownian motion are given by Dasgupta & Kallianpur and Shiryaev with continuous time trading. In the context of discrete time trading arbitrage is discussed in the Ph.D. thesis of Cheridito.

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- Most arbitrage opportunities are based on Riemann-Stieltjes type of understanding of the stochastic integrals. Several authors suggested that one should use divergence [Skorohod] integrals to avoid arbitrage possibilities. However, to give an economical meaning to divergence integrals is difficult, or even impossible.

Mixed fractional Brownian motion

- Consider $X = W + B^H$, where W and B^H are independent. Then X is not a semimartingale with respect to $\mathbb{F}^W \vee \mathbb{F}^{B^H}$; The quadratic variation of X is the same as the quadratic variation of W; X is a semimartingale with respect to $\mathbb{F}^X$, if and only if $H > \frac{3}{4}$ [Cheridito].

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- The existence of quadratic variation implies that

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\begin{aligned}
F(t, X_t) &= F(0,0) + \int_0^t F_t(s, X_s)ds + \int_0^t F_x(s, X_s)dX_s \\
&\quad + \frac{1}{2}\int_0^t F_{xx}(s, X_s)ds.
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  The integral is defined as a limit of forward sums.
- Consider now the mixed process X as the source of the randomness: $dS_t = S_t(\mu dt + \sigma dX_t)$ and hence

$$S_t = s\exp\{\sigma X_t + \mu t - \frac{1}{2}\sigma^2 t\}.$$

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Mixed fractional Brownian motion

- We can now repeat the replication arguments in the classical Black & Scholes model using the PDE approach and we obtain the surprising fact that *the replication price in this mixed model is the same as the replication price in the classical Black & Scholes model!* This was first observed by Kloeden and Schoenmakers.

Mixed fractional Brownian motion

- We can now repeat the replication arguments in the classical Black & Scholes model using the PDE approach and we obtain the surprising fact that *the replication price in this mixed model is the same as the replication price in the classical Black & Scholes model!* This was first observed by Kloeden and Schoenmakers.
- So there seems to be a paradox here: if $H \leq \frac{3}{4}$ for the fractional component B^H there are arbitrage possibilities, but the replication price with continuous trading is the same as in the classical Black & Scholes model.

A discounted *market model* is a five-tuple $(\Omega, \mathcal{F}, S, \mathbb{F}, \mathbf{P})$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a filtered probability space satisfying the usual conditions and $S = (S_t)_{0 \le t \le T}$ is an $\mathcal{F}_t$-progressively measurable positive quadratic variation process with continuous paths starting at $s \in \mathbb{R}$. The constants T and s are fixed.

We assume that our model has the following property:

given any nonnegative continuous function η with $\eta(0) = s$ and any $\epsilon > 0$

$$\mathbf{P}(\{\omega;\ \|S(\omega) - \eta\|_\infty < \epsilon\}) > 0 \tag{1}$$

Given a continuous positive function $\sigma(t,x)$ we define a *model class* by

$$\begin{aligned}\mathcal{M}_\sigma \quad = \quad & \big\{(\Omega,\mathcal{F},S,\mathbb{F},\mathbf{P});\ (\Omega,\mathcal{F},S,\mathbb{F},\mathbf{P}) \text{ is a discounted market model} \\ & \quad \text{satisfying (1) and } d\langle S\rangle_t = \sigma(t,S_t)dt\ P-a.s.\big\}\end{aligned}$$

We will also restrict the possible strategies. In the classical Black & Scholes pricing model the only restriction to strategies is the fact that we do not allow doubling strategies. Here we will restrict more. But we shall still have enough strategies to hedge all practically relevant options.

$g : [0, T] \times \mathcal{C}_{s,+}([0, T]) \to \mathbb{R}$ is a *hindsight factor* if

1. for every $0 \leq t \leq T$ $g(t, \eta) = g(t, \tilde{\eta})$ whenever $\eta(u) = \tilde{\eta}(u)$ for all $0 \leq u \leq t$;

Replication, arbitrage and non-semimartingales

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3. $$\left| \int_0^t f(u) dg(u, \eta) - \int_0^t f(u) dg(u, \tilde{\eta}) \right| \le K \max_{0 \le r \le t} |f(r)| \cdot \|\eta - \tilde{\eta}\|_\infty \tag{2}$$

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E.g. the running maximum, minimum, and average of the stock prices are hindsight factors.

Suppose hindsight factors $g_1, \ldots, g_m$ and a function $\varphi : [0, T] \times \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ are given.

We shall consider strategies of the form

$$\Phi_t = \varphi(t, S_t, g_1(t, S), \ldots, g_m(t, S)). \tag{3}$$

Here Φ_t denotes the number of stocks held by an investor.

Hence, the wealth process corresponding to the strategy Φ is

$$V_t(\Phi, v_0; S) = v_0 + \int_0^t \Phi_s dS_s \tag{4}$$

where $v_0 \in \mathbb{R}$ denotes the investor's initial capital.

Recall that the stochastic integral is defined as a limit of forward sums.

Next we have to specify conditions on φ. We first state a result on absence of arbitrage under the smoothness condition $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}_+ \times \mathbb{R}^m)$.

Φ is supposed to be nds-admissible in the classical sense, i.e. there is a constant $a > 0$ such that for all $0 \leq t \leq T$

$$\int_0^t \Phi_u dS_u \geq -a; \; \mathbf{P} - a.s.$$

A strategy fulfilling these conditions is called a *smooth allowed strategy*.

Replication, arbitrage and non-semimartingales

Smooth no-arbitrage theorem

- Let $(\Omega, \mathcal{F}, S, \mathbb{F}, \mathbf{P}) \in \mathcal{M}_\sigma$ and suppose Φ smooth allowed.

- Let $(\Omega, \mathcal{F}, S, \mathbb{F}, \mathbf{P}) \in \mathcal{M}_\sigma$ and suppose Φ smooth allowed.
- Then Φ cannot be an arbitrage in the model $(\Omega, \mathcal{F}, S, \mathbb{F}, \mathbf{P})$ provided one model $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, \tilde{\mathbb{F}}, \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ admits an equivalent local martingale measure.

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- For example, the model, where the mixed process $X = W + B^H$ is the driving process, and $H \in (\frac{1}{2}, 1)$ does not admit arbitrage with allowed smooth strategies.

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- For example, the model, where the mixed process $X = W + B^H$ is the driving process, and $H \in (\frac{1}{2}, 1)$ does not admit arbitrage with allowed smooth strategies.
- It is known, however, in the classical Black-Scholes model that the smoothness condition $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}_+ \times \mathbb{R}^m)$ is too restrictive to contain hedges even for vanilla options.

Our aim is to extend allowed strategies such that the new class

- contains the natural class of smooth strategies depending on the spot and hindsight factors, i.e. Φ is of form (3) with $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}_+ \times \mathbb{R}^m)$;

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- is sufficiently large to contain hedges for relevant vanilla and exotic options;

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- is sufficiently small to guarantee the absence of arbitrage for the extended class of strategies.

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- is sufficiently large to contain hedges for relevant vanilla and exotic options;
- is sufficiently small to guarantee the absence of arbitrage for the extended class of strategies.

All this is possible to establish, but that would be somewhat technical. We shall not give the details in this talk.

Consider next replication and absence of arbitrage in a class $\mathcal{M}_\sigma$.

- Every model in $\mathcal{M}_\sigma$ is free of arbitrage with allowed strategies provided one admits an equivalent local martingale measure.

Consider next replication and absence of arbitrage in a class $\mathcal{M}_\sigma$.

- Every model in $\mathcal{M}_\sigma$ is free of arbitrage with allowed strategies provided one admits an equivalent local martingale measure.
- Suppose G is a continuous functional on $\mathcal{C}_{s,+}([0,T])$ and in some model $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, \tilde{\mathbb{F}}, \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ there is an allowed strategy $\tilde{\Phi}^*_t = \varphi^*(t, \tilde{S}_t, g_1(t, \tilde{S}), \ldots, g_m(t, \tilde{S}))$ and an initial wealth v_0 such that
$$V_T(\tilde{\Phi}^*, v_0; \tilde{S}) = G(\tilde{S}) \quad \tilde{\mathbf{P}} - a.s.$$
Then in every model $(\Omega, \mathcal{F}, S, \mathbb{F}, \mathbf{P}) \in \mathcal{M}_\sigma$ the allowed strategy $\varphi^*(t, S_t, g_1(t, S), \ldots, g_m(t, S))$ replicates the payoff $G(S)$ at terminal time T $\mathbf{P}$-almost surely and with initial capital v_0.

Concluding remarks

Replication, summary

It has been known that for some pricing models the replication of certain options is the same as in the case of classical Black & Scholes pricing model.

We have extended this to a rather big class of pricing models and strategies.

The class of allowed strategies is big enough to replicate standard options, and small enough to exclude arbitrage.

The replication procedure is the same for each model in a model class!

Concluding remarks

Volatility

It is well known that the implied volatility and the historical volatility do not agree. But if the driving process is mixed fractional, this is clear:

The hedging price depends on the quadratic variation of the stock price S, but the historical volatility is estimated as the variance of the log-returns. These are different notions.

Deviations from Gaussianity

There is a lot of evidence that the log-returns are not Gaussian. By adding a zero-energy process to Brownian motion we do not change the replicating portfolio, but we have a full panorama to change the distributional properties of the stock prices.

Concluding remarks

Irrelevance of probability

By setting (W, B^H) to be jointly Gaussian, say, with suitable covariance structure can have any autocorrelelation we want in the mixed model. However, the hedging prices are not affected. So, in option pricing the probabilistic structure of the log-returns is irrelevant!

References

Föllmer (1981): Calcul d'Itô sans probabilités

Schoenmakers, Kloeden (1999): Robust Option Replication for a Black-Scholes Model Extended with Nondeterministic Trends

Russo, Vallois (1993): Forward, backward and symmetric stochastic integration

Sottinen, Valkeila (2003): On arbitrage and replication in the Fractional Black-Scholes pricing model.

This talk: Bender, Sottinen, Valkeila (2006): On Replication and Absence of Arbitrage in Non-Semimartingale Models.