ANALYSIS OF FINANCIAL TIME SERIES

Nonlinear Univariate
and
Linear Multivariate
Time Series

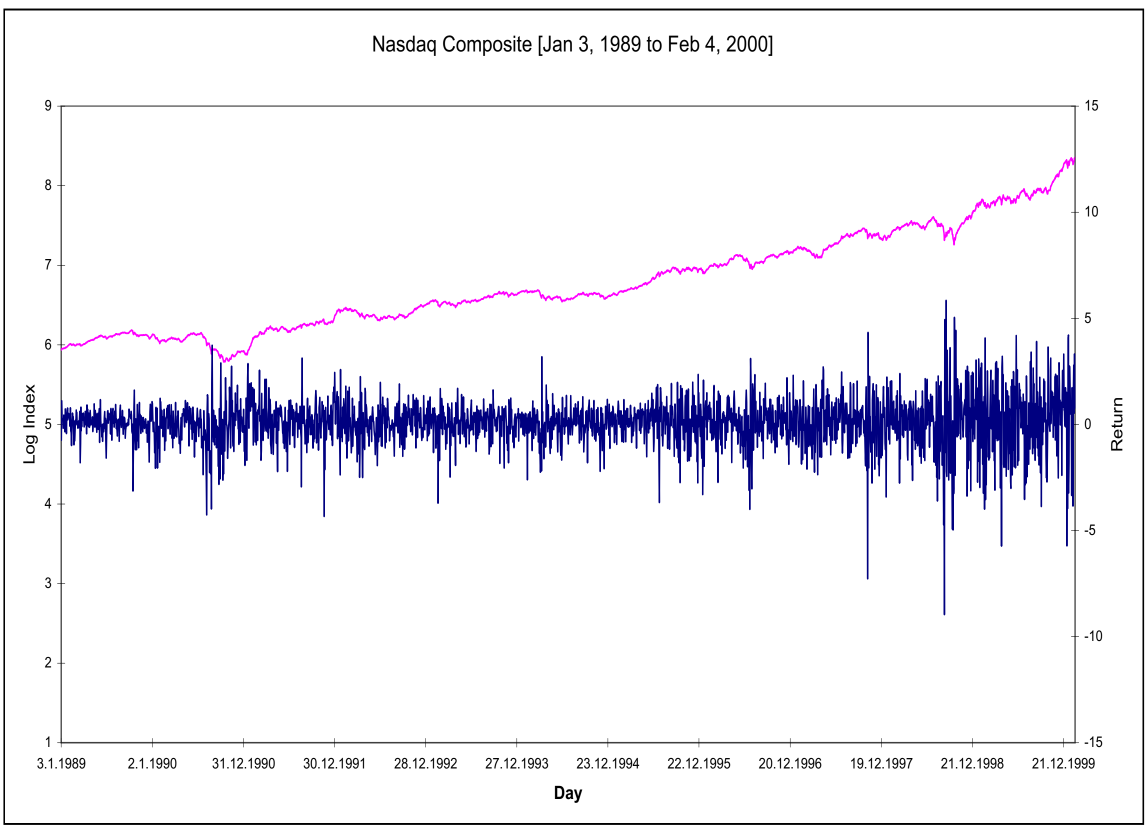
Seppo Pynnönen and
Bernd Pape, 2005*

*University of Vaasa, Dep. of Mathematics and Statistics, P.O. Box 700, 65100 Vaasa, phone: (06) 324 8277 email: bepa@uwasa.fi, URL: http://www.uwasa.fi/~bepa/

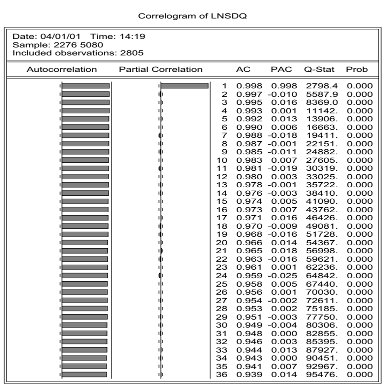
1. Nonlinear Univariate Times Series

1.1 *Background*

Example. Consider the following daily close-to-close Nasdaq composite share index values [January 3, 1989 to February 4, 2000]



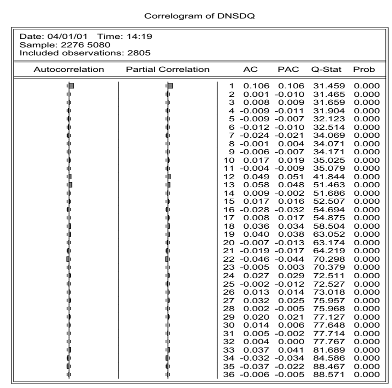
Below are autocorrelations of the log-index. Obviously the persistence of autocorrelations indicate that the series is integrated.† The autocorrelations of the return series suggest that the returns are stationary with statistically significant first order autocorrelation.



Correlogram of LNSDQ

Date: 04/01/01 Time: 14:19
Sample: 2276 5080
Included observations: 2805

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.998	0.998	2798.4	0.000
		2	0.997	-0.010	5587.9	0.000
		3	0.995	0.016	8369.0	0.000
		4	0.993	0.001	11142.	0.000
		5	0.992	0.013	13906.	0.000
		6	0.990	0.006	16663.	0.000
		7	0.988	-0.018	19411.	0.000
		8	0.987	-0.001	22151.	0.000
		9	0.985	-0.011	24882.	0.000
		10	0.983	0.007	27605.	0.000
		11	0.981	-0.019	30319.	0.000
		12	0.980	0.003	33025.	0.000
		13	0.978	-0.001	35722.	0.000
		14	0.976	-0.003	38410.	0.000
		15	0.974	0.005	41090.	0.000
		16	0.973	0.007	43762.	0.000
		17	0.971	0.016	46426.	0.000
		18	0.970	-0.009	49081.	0.000
		19	0.968	-0.016	51728.	0.000
		20	0.966	0.014	54367.	0.000
		21	0.965	0.018	56998.	0.000
		22	0.963	-0.016	59621.	0.000
		23	0.961	0.001	62236.	0.000
		24	0.959	-0.025	64842.	0.000
		25	0.958	0.005	67440.	0.000
		26	0.956	0.001	70030.	0.000
		27	0.954	-0.002	72611.	0.000
		28	0.953	0.002	75185.	0.000
		29	0.951	-0.003	77750.	0.000
		30	0.949	-0.004	80306.	0.000
		31	0.948	0.000	82855.	0.000
		32	0.946	0.003	85395.	0.000
		33	0.944	0.013	87927.	0.000
		34	0.943	0.000	90451.	0.000
		35	0.941	0.007	92967.	0.000
		36	0.939	0.014	95476.	0.000



Correlogram of DNSDQ

Date: 04/01/01 Time: 14:19
Sample: 2276 5080
Included observations: 2805

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.106	0.106	31.459	0.000
		2	0.001	-0.010	31.465	0.000
		3	0.008	0.009	31.659	0.000
		4	-0.009	-0.011	31.904	0.000
		5	-0.009	-0.007	32.123	0.000
		6	-0.012	-0.010	32.514	0.000
		7	-0.024	-0.021	34.069	0.000
		8	-0.001	0.004	34.071	0.000
		9	-0.006	-0.007	34.171	0.000
		10	0.017	0.019	35.025	0.000
		11	-0.004	-0.009	35.079	0.000
		12	0.049	0.051	41.844	0.000
		13	0.058	0.048	51.463	0.000
		14	0.009	-0.002	51.686	0.000
		15	0.017	0.016	52.507	0.000
		16	-0.028	-0.032	54.694	0.000
		17	0.008	0.017	54.875	0.000
		18	0.036	0.034	58.504	0.000
		19	0.040	0.038	63.052	0.000
		20	-0.007	-0.013	63.174	0.000
		21	-0.019	-0.017	64.219	0.000
		22	-0.046	-0.044	70.298	0.000
		23	-0.005	0.003	70.379	0.000
		24	0.027	0.029	72.511	0.000
		25	-0.002	-0.012	72.527	0.000
		26	0.013	0.014	73.018	0.000
		27	0.032	0.025	75.957	0.000
		28	0.002	-0.005	75.968	0.000
		29	0.020	0.021	77.127	0.000
		30	0.014	0.006	77.648	0.000
		31	0.005	-0.002	77.714	0.000
		32	0.004	0.000	77.767	0.000
		33	0.037	0.041	81.689	0.000
		34	-0.032	-0.034	84.586	0.000
		35	-0.037	-0.022	88.467	0.000
		36	-0.006	-0.005	88.571	0.000

Figure. Nasdaq Composite index autocorrelations for log levels and log differences (returns)

†**Definition**. Time series y_t, $t = 1, \ldots, T$ is covariance stationary if

$$\begin{aligned} \mathrm{E}[y_t] &= \mu, \text{ for all } t \\ \mathrm{cov}[y_t, y_{t+k}] &= \gamma_k, \text{ for all } t \\ \mathrm{var}[y_t] &= \gamma_0 \ (< \infty), \text{ for all } t \end{aligned}$$

Any series that are not stationary are said to be nonstationary.

Definition Times series y_t is said to be integrated of order d, denoted as $y_t \sim I(d)$, if $\Delta^d y_t$ is stationary. Note that if y_t is stationary then $y_t = \Delta^0 y_t$. Thus for short a stationary series is denoted as $y_t \sim I(0)$, i.e., integrated of order zero.

Below are results after fitting an AR(1) and an MA(1) model to the return series

Table. AR(1) estimates.

Dependent Variable: DNSDQ
Method: Least Squares
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 2 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.086126	0.023048	3.736845	0.0002
AR(1)	0.105933	0.018782	5.640001	0.0000

R-squared	0.011221	Mean dependent var	0.086119
Adjusted R-squared	0.010868	S.D. dependent var	1.097336
S.E. of regression	1.091357	Akaike info criterion	3.013434
Sum squared resid	3338.542	Schwarz criterion	3.017668
Log likelihood	-4224.341	F-statistic	31.80961
Durbin-Watson stat	1.997947	Prob(F-statistic)	0.000000
Inverted AR Roots	.11		

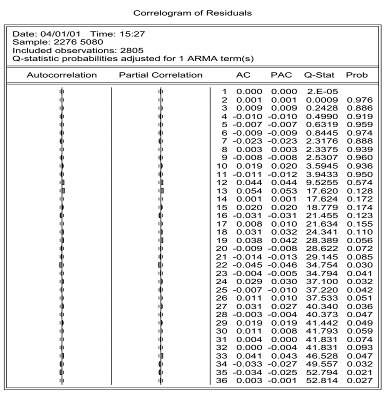
Table. MA(1) estimates

Dependent Variable: DNSDQ
Method: Least Squares
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 4 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.086153	0.022811	3.776796	0.0002
MA(1)	0.107093	0.018779	5.702685	0.0000

R-squared	0.011323	Mean dependent var	0.086119
Adjusted R-squared	0.010970	S.D. dependent var	1.097336
S.E. of regression	1.091301	Akaike info criterion	3.013331
Sum squared resid	3338.198	Schwarz criterion	3.017565
Log likelihood	-4224.196	F-statistic	32.10153
Durbin-Watson stat	2.000051	Prob(F-statistic)	0.000000
Inverted MA Roots	-.11		

Both models give virtually equally good fit, MA(1) only just marginally better. The residual autocorrelations and related Q-statistics indicate no further autocorrelation left to the series.

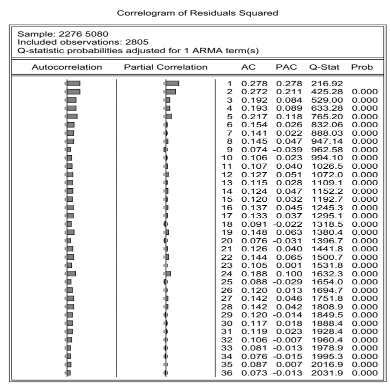


Correlogram of Residuals

Date: 04/01/01 Time: 15:27
Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.000	0.000	2.E-05	
2	0.001	0.001	0.0009	0.976
3	0.009	0.009	0.2428	0.886
4	-0.010	-0.010	0.4990	0.919
5	-0.007	-0.007	0.6319	0.959
6	-0.009	-0.009	0.8445	0.974
7	-0.023	-0.023	2.3176	0.888
8	0.003	0.003	2.3375	0.939
9	-0.008	-0.008	2.5307	0.960
10	0.019	0.020	3.5945	0.936
11	-0.011	-0.012	3.9433	0.950
12	0.044	0.044	9.5255	0.574
13	0.054	0.053	17.620	0.128
14	0.001	0.001	17.624	0.172
15	0.020	0.020	18.779	0.174
16	-0.031	-0.031	21.455	0.123
17	0.008	0.010	21.634	0.155
18	0.031	0.032	24.341	0.110
19	0.038	0.042	28.389	0.056
20	-0.009	-0.008	28.622	0.072
21	-0.014	-0.013	29.145	0.085
22	-0.045	-0.046	34.754	0.030
23	-0.004	-0.005	34.794	0.041
24	0.029	0.030	37.100	0.032
25	-0.007	-0.010	37.220	0.042
26	0.011	0.010	37.533	0.051
27	0.031	0.027	40.340	0.036
28	-0.003	-0.004	40.373	0.047
29	0.019	0.019	41.442	0.049
30	0.011	0.008	41.793	0.059
31	0.004	0.000	41.831	0.074
32	0.000	-0.004	41.831	0.093
33	0.041	0.043	46.528	0.047
34	-0.033	-0.027	49.557	0.032
35	-0.034	-0.025	52.794	0.021
36	0.003	-0.001	52.814	0.027

Figure. Autocorrelations of the squared MA(1) residuals



Correlogram of Residuals Squared

Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.278	0.278	216.92	
2	0.272	0.211	425.28	0.000
3	0.192	0.084	529.00	0.000
4	0.193	0.089	633.28	0.000
5	0.217	0.118	765.20	0.000
6	0.154	0.026	832.06	0.000
7	0.141	0.022	888.03	0.000
8	0.145	0.047	947.14	0.000
9	0.074	-0.039	962.58	0.000
10	0.106	0.023	994.10	0.000
11	0.107	0.040	1026.5	0.000
12	0.127	0.051	1072.0	0.000
13	0.115	0.028	1109.1	0.000
14	0.124	0.047	1152.2	0.000
15	0.120	0.032	1192.7	0.000
16	0.137	0.045	1245.3	0.000
17	0.133	0.037	1295.1	0.000
18	0.091	-0.022	1318.5	0.000
19	0.148	0.063	1380.4	0.000
20	0.076	-0.031	1396.7	0.000
21	0.126	0.040	1441.8	0.000
22	0.144	0.065	1500.7	0.000
23	0.105	0.001	1531.8	0.000
24	0.188	0.100	1632.3	0.000
25	0.088	-0.029	1654.0	0.000
26	0.120	0.013	1694.7	0.000
27	0.142	0.046	1751.8	0.000
28	0.142	0.042	1808.9	0.000
29	0.120	-0.014	1849.5	0.000
30	0.117	0.018	1888.4	0.000
31	0.119	0.023	1928.4	0.000
32	0.106	-0.007	1960.4	0.000
33	0.081	-0.013	1978.9	0.000
34	0.076	-0.015	1995.3	0.000
35	0.087	0.007	2016.9	0.000
36	0.073	-0.013	2031.9	0.000

Figure. Autocorrelations of the squared MA(1) residuals

Recall, however, that uncorrelatedness does not imply independence. The autocorrelations of the squared residuals strongly suggest that there is still left time dependence in the series. Because squared residuals are the building blocks of the variance of the series (recall: the sample variance of n observations $x_1, x_2, \ldots x_n$ is $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)$), the results suggest that the variation (volatility) of the series is time dependent. This leads to the so called ARCH-family of models.

1.2 *ARCH-models*[‡]

The general setup for ARCH models is

$$y_t = \mathbf{x}_t'\beta + u_t$$

with $\mathbf{x}_t = (x_{1t}, x_{2t}, \ldots, x_{pt})'$, $\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$, $t = 1, \ldots, T$, and

$$u_t|\mathcal{F}_{t-1} \sim N(0, h_t),$$

where $\mathcal{F}_t$ is the information available at time t (usually the past values of u_t; $u_1, \ldots, u_{t-1}$), and

$$h_t = \mathrm{var}(u_t|\mathcal{F}_{t-1}) = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2.$$

Furthermore, it is assumed that $\omega > 0$ and $\alpha_i \geq 0$ for all i (to ensure positive variance), and $\alpha_1 + \cdots + \alpha_q < 1$ (to get stationarity). For short the model is denoted $u_t \sim$ ARCH(q).

[‡]The inventor of this modeling approach is Robert F. Engle (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50, 987–1008.

Properties of ARCH-processes

Consider (for the sake of simplicity) ARCH(1) process

$$h_t = \omega + \alpha u_{t-1}^2$$

with $\omega > 0$ and $0 \leq \alpha < 1$ and $u_t | u_{t-1} \sim N(0, h_t)$.

(a) u_t is white noise

(i) Constant mean (zero):

$$\mathsf{E}[u_t] = \mathsf{E}[\underbrace{\mathsf{E}_{t-1}[u_t]}_{=0}] = \mathsf{E}[0] = 0.$$

Note $\mathsf{E}_{t-1}[u_t] = \mathsf{E}[u_t|\mathcal{F}_{t-1}]$, the conditional expectation given information up to time $t-1$.[§]

[§] The law of iterated expectations: Consider time points $t_1 < t_2$ such that $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$, then for any $t_3 > t_2$

$$\mathsf{E}_{t_1}\left[E_{t_2}[u_{t_3}]\right] = \mathsf{E}\left[\mathsf{E}[u_{t_3}|\mathcal{F}_{t_2}]|\mathcal{F}_{t_1}\right] = \mathsf{E}[u_{t_3}|\mathcal{F}_{t_1}] = \mathsf{E}_{t_1}[u_{t_3}],$$

which impies in particular $\mathsf{E}\left[E_{t_2}[u_{t_3}]\right] = \mathsf{E}[u_{t_3}]$.

(ii) Constant variance: Using again the law of iterated expectations, we get

$$
\begin{aligned}
\sigma_u^2 := \mathrm{var}[u_t] &= \mathsf{E}[u_t^2] = \mathsf{E}\left[\mathsf{E}_{t-1}[u_t^2]\right] \\
&= \mathsf{E}[h_t] = \mathsf{E}[\omega + \alpha u_{t-1}^2] \\
&= \omega + \alpha \mathsf{E}[u_{t-1}^2] = \omega + \alpha \mathsf{E}[h_{t-1}] \\
&= \omega + \alpha(\omega + \alpha \mathsf{E}[u_{t-2}^2]) \\
&= \omega(1+\alpha) + \alpha^2 \mathsf{E}[u_{t-2}^2] \\
&\vdots \\
&= \omega(1 + \alpha + \alpha^2 + \cdots + \alpha^n) \\
&\quad + \underbrace{\alpha^{n+1}\mathsf{E}[u_{t-n-1}^2]}_{\to 0,\ \text{as } n\to\infty} \\
&= \omega\left(\lim_{n\to\infty} \textstyle\sum_{i=0}^{n} \alpha^i\right) \\
&= \frac{\omega}{1-\alpha}.
\end{aligned}
$$

(iii) Autocovariances: $\mathrm{cov}(u_t, u_{t+k}) = 0$ for all $k \neq 0$, because using again the law of iterated expectations we get

$$
\begin{aligned}
\mathrm{cov}(u_t, u_{t+k}) &= \mathsf{E}[u_t u_{t+k}] = \mathsf{E}(\mathsf{E}_{t+k-1}(u_t u_{t+k})) \\
&= \mathsf{E}[u_t \mathsf{E}_{t+k-1}(u_{t+k})) = \mathsf{E}(u_t \cdot 0) \\
&= 0.
\end{aligned}
$$

(b) The unconditional distribution of u_t is symmetric, but nonnormal.

(i) Skewness: Exercise, show that $\mathrm{E}[u_t^3] = 0$.

(ii) Kurtosis: Exercise, show that under the assumption $u_t|u_{t-1} \sim N(0, h_t)$, and that $\alpha < \sqrt{1/3}$, the kurtosis

$$\mathrm{E}[u_t^4] = 3\frac{\omega^2}{(1-\alpha)^2} \cdot \frac{1-\alpha^2}{1-3\alpha^2}.$$

Because $(1-\alpha^2)/(1-3\alpha^2) > 1$ we have that

$$\mathrm{E}[u_t^4] > 3\frac{\omega^2}{(1-\alpha)^2} = 3[\mathrm{var}(u_t)]^2 = 3\sigma_u^4,$$

we find that the kurtosis of the unconditional distribution exceed that what it would be, if u_t were normally distributed. Thus the unconditional distribution of u_t is nonnormal and has fatter tails than a normal distribution with variance equal to $\sigma_u^2 = \omega/(1-\alpha)$.

(c) Analogy with AR-processes

Consider again the specification of the variance in a general ARCH(q) process:

$$h_t = \text{var}(u_t|\mathcal{F}_{t-1}) = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2.$$

This reminds essentially of an AR(q) process for the squared residuals, because defining $\nu_t = u_t^2 - h_t$, we can write

$$u_t^2 = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 + \nu_t.$$

The expected value of ν_t is zero, but var(ν_t) is time dependent (see below), implying that this is not a stationary process in the sense defined above. This implies that the conventional estimation procedure in AR-estimation does not produce optimal results here.

(d) Standardized variables

Write

$$z_t = \frac{u_t}{\sqrt{h_t}}$$

then $z_t | \mathcal{F}_{t-1} \sim \text{N}(0, 1)$ implying $z_t \sim \text{NID}(0, 1)$, i.e., normally and independently distributed. Thus we can always write

$$u_t = z_t \sqrt{h_t},$$

where the z_t are independent standard normal random variables (strict white noise). This gives us a useful device to check after fitting an ARCH model the adequacy of the specification: Check the autocorrelations of the squared standardized series.

Returning to the analogy with AR-processes, we note that using standardized variables we may write

$$\nu_t = u_t^2 - h_t = z_t^2 h_t - h_t = h_t(z_t^2 - 1).$$

The factor on the right hand side is identically and independently distributed (iid), that is, in particular independent of h_t, with zero mean and variance 2 (because z_t^2 is iid $\chi^2(1)$). So ν_t has zero mean too, but time-dependent variance, because h_t is time-dependent.

Estimation of ARCH models

Given the model

$$y_t = \mathbf{x}_t'\beta + u_t$$

with $u_t|\mathcal{F}_{t-1} \sim N(0, h_t)$, we have $y_t|\{\mathbf{x}_t, \mathcal{F}_{t-1}\} \sim N(\mathbf{x}_t'\beta, h_t)$, $t = 1, \ldots, T$ with conditional pdf

$$f(y_t|\mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi h_t}} e^{-\frac{(y_t - \mathbf{x}_t'\beta)^2}{2h_t}}$$

The log-likelihood function is then $\ell(\theta) = \sum_{t=1}^{T} \ell_t(\theta)$ with

$$\ell_t(\theta) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log h_t - \frac{1}{2}(y_t - \mathbf{x}_t'\beta)^2/h_t,$$

where $\theta = (\beta', \omega, \alpha)'$.

The maximum likelihood (ML) estimate $\widehat{\theta}$ is the value maximizing the likelihood function, i.e.,

$$\ell(\widehat{\theta}) = \max_{\theta} \ell(\theta).$$

The maximization is accomplished by numerical methods.

Note: OLS estimates of the regression parameters are inefficient (unreliable) compared to the ML estimates.

Variance Forecasting with ARCH models

Consider again for simplicity the ARCH(1) process with $h_t = \omega + \alpha u_{t-1}^2$. The expected value at time t of the variance k time steps ahead is

$$\mathsf{E}_t(u_{t+k}^2) = \mathsf{E}_t\mathsf{E}_{t+k-1}(u_{t+k}^2) = E_t(h_{t+k})\omega + \alpha \mathsf{E}_t(u_{t+k-1}^2).$$

We apply this recursively to get:

$$\begin{aligned}
\mathsf{E}_t(u_{t+1}^2) &= \omega + \alpha \mathsf{E}_t(u_t^2) = \omega + \alpha u_t^2 = h_{t+1} \\
\mathsf{E}_t(u_{t+2}^2) &= \omega + \alpha \mathsf{E}_t(u_{t+1}^2) = \omega + \alpha h_{t+1} \\
\mathsf{E}_t(u_{t+3}^2) &= \omega + \alpha \mathsf{E}_t(u_{t+2}^2) = \omega + \alpha(\omega + \alpha h_{t+1}) \\
&= \omega(1+\alpha) + \alpha^2 h_{t+1} \\
&\vdots \\
\mathsf{E}_t(u_{t+k}^2) &= \omega(1 + \alpha + \ldots + \alpha^{k-2}) + \alpha^{k-1} h_{t+1} \\
&= \omega \frac{1-\alpha^{k-1}}{1-\alpha} + \alpha^{k-1} h_{t+1} \\
&= \frac{\omega}{1-\alpha} - \frac{\omega \alpha^{k-1}}{1-\alpha} + \alpha^{k-1} h_{t+1} \\
&= \frac{\omega}{1-\alpha} + \alpha^{k-1}\left(h_{t+1} - \frac{\omega}{1-\alpha}\right) \\
&= \sigma_u^2 + \alpha^{k-1}(h_{t+1} - \sigma_u^2) \stackrel{k\to\infty}{\longrightarrow} \sigma_u^2.
\end{aligned}$$

We note that for long forecast horizons k, the forecasts of the conditional variance h_{t+k} approach the unconditional variance $\sigma_u^2 = \omega/(1-\alpha)$.