Mathematics of Financial Derivatives: Exercise Sheet 4

1. In order to justify the shorthand notation $dW_1\,dW_2 = 0$, show that the Itó integral of two independent Wiener processes W_1 and W_2 is zero, that is:
$$\int_0^t dW_1\,dW_2 = 0.$$
Hint: Partition the time interval $[0,t]$ into n subintervals of equal length (t/n) and consider $\mathrm{E}(I_n(t)^2)$ in the limit $n \to \infty$, with elementary integrals $I_n(t) = \sum_{i=1}^n (W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)})(W_{t_i}^{(2)} - W_{t_{i-1}}^{(2)})$.

2. a) Using Ito's formula, show that
$$S_t = (W_t + \sqrt{S_0})^2$$
is the solution of the stochastic differential equation
$$dS_t = dt + 2\sqrt{S_t}dW_t.$$
Hint: A reasonable start off guess for $\int dS_t$ is $F(S_t,t) = {S_t}^{1/2}$. Apply Itos formula and integrate on both sides.

   b) Cross-check your result by applying Itos formula to S_t above.

3. a) Using Ito's formula, show that
$$S_t = e^{-\mu t} S_0 + e^{-\mu t} \int_0^t e^{\mu s} \sigma\, dW_s$$
is the solution of the Ornstein-Uhlenbeck SDE
$$dS_t = -\mu S_t dt + \sigma dW_t,$$
where μ and σ are constants, and W_t is a standard Wiener process. *Hint: A reasonable start off guess for $\int dS_t$ is $F(S_t,t) = e^{\mu t} S_t$. Apply Itos formula and integrate on both sides.*

   b) Cross-check your result by applying Itos formula to S_t above. *Hint: You may write the process as $S_t = e^{-\mu t}(S_0 + X_t)$ with $X_t = \int_0^t e^{\mu s}\sigma\, dW_s$. Note that $X_t = \int_0^t f(s) dW_s$ implies $dX_t = f(t) dW_t$.*

4. The goal of this exercise is to show that the Black-Scholes call option formula
$$F(S_t,t) = S_t N(d_1) - X e^{-r\tau} N(d_2),$$

with

$$d_1 = \frac{\log(S_t/X) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

$$d_2 = \frac{\log(S_t/X) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau},$$

$$\tau = T - t, \quad \text{and} \quad N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{1}{2}x^2}\, dx, \quad i = 1, 2$$

satisfies the Black-Scholes PDE:

$$-r\,F + r\,F_s S + F_t + \frac{1}{2}\sigma^2 F_{ss} S^2 = 0.$$

a) Calculate the partial derivatives

$$\frac{\partial d_1}{\partial S}, \quad \frac{\partial d_2}{\partial S}, \quad \frac{\partial d_1}{\partial \tau}, \quad \text{and} \quad \frac{\partial d_2}{\partial \tau}.$$

b) Show that $S_t N'(d_1) = X e^{-r\tau} N'(d_2)$ where $N'(d_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_i^2}$ for $i = 1, 2$ by the fundamental theorem of calculus.
Hint: It is probably easiest to start from the right side.
Use $d_2 = d_1 - \sigma\sqrt{\tau}$ and $X = e^{\ln X}$.

c) Show that

$$F_s = N(d_1) \quad \text{and} \quad F_{ss} = \frac{N'(d_1)}{\sigma S \sqrt{\tau}}.$$

Hint: Note that F depends upon S also indirectly through $N(d_1)$ and $N(d_2)$. Therefore you need to use your result obtained in b).

d) Show that

$$F_t = -rXe^{-r\tau} N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{\tau}}.$$

Hint: Note that

$$F_t = \frac{\partial F}{\partial t} = \frac{\partial F}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -\frac{\partial F}{\partial \tau}.$$

You will several times have to use your result obtained in b).

e) Show that F satisfies the Black-Scholes PDE:

$$-r\,F + r\,F_s S + F_t + \frac{1}{2}\sigma^2 F_{ss} S^2 = 0.$$