Statistical Formulas and Tables
for use with
STAT1010: Riippuvuusanalyysi

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Measures of Contingency

Pearson's χ^2:

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}, \tag{1}$$

where
f_{ij} are the observed frequencies in cell (i,j), and
$e_{ij} = \frac{f_{i\bullet} f_{\bullet j}}{n}$ with $f_{i\bullet} = \sum_j f_{ij}$ and $f_{\bullet j} = \sum_i f_{ij}$
are the expected frequencies in cell (i,j) under independence
of r row variables and s column variables.

Contingency Coefficient:

$$C = \sqrt{\frac{\chi^2}{\chi^2 + n}} \qquad \text{with maximum} \qquad C_{\max} = \sqrt{\frac{k-1}{k}}, \tag{2}$$

where k is the smaller number of the rows r and columns s.

Cramer's V:

$$V = \sqrt{\frac{\chi^2}{\chi^2_{\max}}}, \qquad \text{where} \qquad \chi^2_{\max} = n(k-1) \tag{3}$$

is the largest possible value of Pearson's χ^2-statistic in a table with r rows, s columns, and n observations ($k = \min(r, s)$).

Spearman's rank correlation ρ_S:

$$\rho_S = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2 - 1)}, \tag{4}$$

where the d_i's are the differences between the ranks of n observation pairs.

Kendall's rank correlation τ:

$$\tau = 1 - \frac{4Q}{n(n-1)}, \tag{5}$$

where Q is the number of discordant pairs within n observation pairs.

χ^2-based Tests

χ^2 independence test:

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s} \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \sim \chi^2((r-1)(s-1)) \text{ under} \tag{6}$$

$$H_0\colon X \text{ and } Y \text{ are statistically independent}$$

Continuity adjustment for two-way tables, that is $r = s = 2$:

$$\chi^2 = \frac{n(|f_{11}f_{22} - f_{12}f_{21}| - n/2)^2}{f_{1\bullet}f_{2\bullet}f_{\bullet 1}f_{\bullet 2}} \sim \chi^2(1) \text{ under } H_0 \tag{7}$$

with f_{ij} observed frequencies in cell (i,j), $f_{i\bullet} = f_{i1} + f_{i2}$, and $f_{\bullet j} = f_{1j} + f_{2j}$.

χ^2-test for equality of proportions:

χ^2 independence test upon the following contingency table,

	sample 1	$\cdots$	sample k	sum
success	n_1p_1	$\cdots$	n_kp_k	$f_{1\bullet} = \sum n_ip_i$
failure	$n_1(1-p_1)$	$\cdots$	$n_k(1-p_k)$	$f_{2\bullet} = \sum n_i(1-p_i)$
sum:	$f_{\bullet 1} = n_1$	$\cdots$	$f_{\bullet k} = n_k$	$n = \sum n_i$

with n_i = size of sample i, and p_i = proportion of success in sample i.

Median test:

χ^2 independence test with expected frequencies $e_{ij} = n_i/2$,
where n_i is the sample size from population i.

McNemar test on paired proportions:

		Sample II: 1	Sample II: 0
Sample I:	1	$a = np_{11}$	$b = np_{10}$
	0	$c = np_{01}$	$d = np_{00}$

$$\chi^2 = \frac{(|b-c| - 1)^2}{b+c} \sim \chi^2(1) \text{ under} \tag{8}$$

H_0: The probability of success in both samples is identical.

Nonparametric Tests

Sign Test:

With n observation pairs (x_i, y_i), the test statistic is T = number of plus signs, where $x_i > y_i \to (+)$ and $x_i < y_i \to (-)$ and pairs with $x_i = y_i$ (ties) are discarded.

$$\text{Under } H_0\colon\ P(X > Y) = \frac{1}{2}, \qquad T \sim \text{Bin}\left(n, \frac{1}{2}\right). \tag{9}$$

$$\text{For } n \to \infty : \qquad Z = \frac{2T - n}{\sqrt{n}} \sim N(0,1) \text{ under } H_0. \tag{10}$$

Wilcoxon Signed-Rank Test:

The hypotheses of the two-sided test for paired observations are

H_0: median of population 1 = median of population 2,
H_1: median of population 1 $\neq$ median of population 2.

The Wilcoxon T statistic is defined as the smaller of the two sums of ranks,

$$T = \min\left[\sum(+), \sum(-)\right],$$

where $\sum(+/-)$ is the sum of the ranks of the positive/negative differences of the observation pairs (x_1, x_2). We reject the null hypothesis of the two-sided test if the computed value of the statistic is less than the $\frac{\alpha}{2}$ critical point from the table. We reject H_0 against the one-sided alternative

H_1: median of population 1 $>$ ($<$) median of population 2

if $\sum(-)$ ($\sum(+)$) is less than the α critical point from the table.

Kruskal-Wallis Test

Let $n_i,\ i = 1, \ldots, k$ denote the sample size from population i and let $n = n_1 + \cdots + n_k$. Defining $R_i,\ i = 1, \ldots, k$ as the sum of ranks from sample k, the Kruskal-Wallis test statistic is

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(n+1) \sim \chi^2(k-1) \qquad \text{under} \tag{11}$$

H_0: All k populations have the same distribution;

as long as each $n_i \geq 5$.

Runs test

The hypotheses of the runs test are:

H_0: The observations are generated randomly,
H_1: The observations are not generated randomly;

with test statistic: R = number of runs. For large sample sizes $n = n_+ + n_-$:

$$z = \frac{R - E(R)}{\sigma_r} \sim N(0,1) \quad \text{under } H_0, \text{ where} \tag{12}$$

$$E(R) = \frac{2n_+n_-}{n} + 1, \qquad \sigma_R = \sqrt{\frac{2n_+n_-(2n_+n_- - n)}{n^2(n-1)}} \tag{13}$$

We reject H_0 at significance level α if $|z| > z(\frac{\alpha}{2})$.

Principles of ANOVA

Consider k independent samples:

$$\begin{gathered} x_{11}, x_{12}, \ldots, x_{1n_1}, \\ \vdots \\ x_{k1}, x_{k2}, \ldots, x_{kn_k}, \end{gathered}$$

where $x_{ij} \sim N(\mu_i, \sigma^2)$ and $n_1 + \cdots + n_k = N$. We wish to test whether all observations come from the same distribution or not.

1) The Sum of Squares Principle

$$\text{grand mean:} \qquad \overline{\overline{x}} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} = \frac{1}{N}\sum_{i=1}^{k} n_i \overline{x_i} \tag{14}$$

$$\text{SST} = \sum_i \sum_j (x_{ij} - \overline{\overline{x}})^2 = (N-1)s^2 \tag{15}$$

$$= \sum_i \overbrace{\sum_j (\overline{x_i} - \overline{\overline{x}})^2}^{n_i(\overline{x_i}-\overline{\overline{x}})^2} + \sum_i \overbrace{\sum_j (x_{ij} - \overline{x_i})^2}^{(n_i-1)s_i^2} \tag{16}$$

$$= \qquad \text{SSB} \qquad + \qquad \text{SSE} \tag{17}$$

where $\overline{x_i}$ and s_i^2 are the mean and the sample variance of the observations in the i'th sample, and s^2 is the sample variance of all observations.

2) Additivity of degrees of freedom

The degrees of freedom of SST are $\text{DFT} = (N-1)$.
The degrees of freedom of SSB are $\text{DFB} = (k-1)$.
The degrees of freedom of SSE are $\text{DFE} = (N-k)$.

$$\text{DFT} = \text{DFB} + \text{DFE}. \tag{18}$$

3) The Mean Squares and the F-test

$$F = \frac{MSB}{MSE} \sim F(k-1, N-k) \quad \text{under} \tag{19}$$

$$H_0\colon \ \mu_1 = \mu_2 = \ldots = \mu_k, \qquad \text{where}$$

$$\text{MSB} = \frac{\text{SSB}}{\text{DFB}} \quad \text{and} \quad \text{MSE} = \frac{\text{SSE}}{\text{DFE}}. \tag{20}$$

4) Estimation of the effects

Fixed effects model: $$a_i = \overline{x_i} - \overline{\overline{x}}, \quad i = 1, 2, \ldots k. \tag{21}$$

Random effects model: $$s_A^2 = \frac{\text{MSB} - \text{MSE}}{n_0} \quad \text{with} \tag{22}$$

$$n_0 = \frac{N^2 - \sum_{i=1}^{k} n_i^2}{N(k-1)}, \tag{23}$$

where $n_0 = n$ in the special case of equal sample sizes $n = N/k$ in all groups.

Contrasts

$$t = \frac{L}{s(L)} \sim t(\text{DFE}) \quad \text{where} \tag{24}$$

$$L = \sum \lambda_i \overline{x_i} \quad \text{and} \quad s(L) = \sqrt{\text{MSE} \sum \frac{\lambda_i^2}{n_i}} \tag{25}$$

$$\text{under } H_0\colon \lambda_1\mu_1 + \lambda_2\mu_2 + \cdots + \lambda_k\mu_k = 0,$$
$$\text{where} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = 0.$$

Two contrasts $\psi_{1/2} = \sum_{i=1}^{k} \lambda_{1/2i}\mu_i$ with $\sum_{i=1}^{k} \lambda_{1/2i} = 0$ are orthogonal if

$$\sum_{i=1}^{k} \frac{\lambda_{1i}\lambda_{2i}}{n_i} = 0. \tag{26}$$

Simultaneous Confidence Intervals

$$CI_{1-\alpha} = [(\overline{x}_i - \overline{x}_j) \pm \text{smallest significant difference}], \tag{27}$$

where the smallest significant differences are:

$$LSD\colon t_{\frac{\alpha}{2}}(\text{DFE}) \cdot s\sqrt{\frac{1}{n_i} + \frac{1}{n_j}} \tag{28}$$

$$\text{Bonferroni: } t_{\frac{\alpha}{2}/\binom{k}{2}}(\text{DFE}) \cdot s\sqrt{\frac{1}{n_i} + \frac{1}{n_j}} \tag{29}$$

$$HSD\colon q_\alpha(k, \text{DFE}) \cdot \frac{s}{\sqrt{n}} = q_\alpha(k, \text{DFE}) \cdot \sqrt{\frac{\text{MSE}}{2}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} \tag{30}$$

with sample sizes n_i and n_j, resp. n for common sample sizes and $s = \sqrt{\text{MSE}}$.

Two-Way ANOVA

Consider taking independent samples from data which is split into $I \times J$ populations according to I levels of factor A and J levels according to factor B. Denote with X_{ijk} the k'th observation of the sample corresponding to level i of factor A and level j of factor B and assume

$$X_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \qquad \epsilon_{ijk} \sim N(0, \sigma^2), \tag{31}$$

$$\text{such that} \qquad \sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = \sum_{i=1}^{I} \sum_{j=1}^{J} (\alpha\beta)_{ij} = 0. \tag{32}$$

ANOVA Table for Two-Way Analysis

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F Ratio
Factor A	SSA	$I-1$	MSA= $\frac{\text{SSA}}{I-1}$	$F = \frac{\text{MSA}}{\text{MSE}}$
Factor B	SSB	$J-1$	MSB= $\frac{\text{SSB}}{J-1}$	$F = \frac{\text{MSB}}{\text{MSE}}$
Interaction	SSAB	$(I-1)(J-1)$	MSAB= $\frac{\text{SSAB}}{(I-1)(J-1)}$	$F = \frac{\text{MSAB}}{\text{MSE}}$
Error	SSE	$N-IJ$	MSE= $\frac{\text{SSE}}{N-IJ}$	
Total	SST	$N-1$		

where N denotes the total number of observations.

Hypothesis Tests in Two-Way ANOVA

Factor A main-effects test:

$$F = \text{MSA}/\text{MSE} \sim F(I-1, N-IJ) \tag{33}$$
$$\text{under } H_0: \quad \alpha_i = 0 \text{ for all } i = 1, \ldots, I$$

Factor B main-effects test:

$$F = \text{MSB}/\text{MSE} \sim F(J-1, N-IJ) \tag{34}$$
$$\text{under } H_0: \quad \beta_j = 0 \text{ for all } j = 1, \ldots, J$$

Test for AB interactions:

$$F = MSAB/MSE \sim F((I-1)(J-1), N-IJ) \tag{35}$$
$$\text{under } H_0: \quad (\alpha\beta)_{ij} = 0 \text{ for all } i = 1, \ldots, I;\ j = 1, \ldots, J$$

ANOVA Table for Randomized Complete Block Design

Consider k treatments, which are randomly assigned to b blocks.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F Ratio
Blocks	SSBL	$b-1$	$\text{MSBL}=\dfrac{\text{SSBL}}{b-1}$	
Treatments	SSTR	$k-1$	$\text{MSTR}=\dfrac{\text{SSTR}}{k-1}$	$F=\dfrac{\text{MSTR}}{\text{MSE}}$
Error	SSE	$(k-1)(b-1)$	$\text{MSE}=\dfrac{\text{SSE}}{(k-1)(b-1)}$	
Total	SST	$kb-1$		

Hypothesis Test in Randomized Complete Block Design

$$F=\frac{\text{MSTR}}{\text{MSE}}\sim F(k-1,(k-1)(b-1)) \quad (36)$$
$$\text{under } H_0: \quad \mu_1=\mu_2=\cdots=\mu_k.$$

Linear Correlation and Regression

Pearsons Linear Correlation Coefficient

$$T=\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\sim t(n-2) \text{ under} \quad (37)$$
$$H_0:\ \rho=0\ (x,y \text{ are linearly independent}).$$

For large n and small r approximately:

$$Z=r\sqrt{n}\sim N(0,1) \text{ under } H_0, \quad (38)$$

where r is the sample linear correlation coefficient and n is the number of observation pairs. The same test may in large samples be used to assess $H_0:\ \rho_S=0$ or $\tau=0$ by replacing r with Spearmans rank correlation r_S or Kendall's τ.

The k-Variable Multiple Linear Regression Model

$$Y=\beta_0+\beta_1X_1+\beta_2X_2+\cdots+\beta_kX_k+\epsilon, \quad (39)$$

Model assumptions:

1. $\epsilon_j\sim NID(0,\sigma^2)$ for all observations $j=1,\dots,n$;
2. The variables X_i are considered fixed quantities (not random variables), that is, the only randomness in Y comes from the error term ϵ.

Sample representation of linear regression in matrix form:

$$\mathbf{y}_{(n\times 1)} = \mathbf{X}_{(n\times(k+1))}\mathbf{b}_{((k+1)\times 1)} + \mathbf{u}_{(n\times 1)}, \quad \text{with}$$

$$\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & x_{2,1} & \cdots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \cdots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,n} & x_{2,n} & \cdots & x_{k,n} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{pmatrix}, \tag{40}$$

$$\mathbf{y}' = (y_1, y_2, \ldots, y_n)' \text{ and } \mathbf{u}' = (u_1, u_2, \ldots, u_n)'.$$

Least-square estimates for $\beta_0, \ldots, \beta_k$:

$$\hat{\beta}_{((k+1)\times 1)} = \mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{41}$$

In the special case of only 1 regressor:

$$b_1 = r_{xy} \cdot \frac{s_y}{s_x} \qquad \text{and} \qquad b_0 = \bar{y} - b_1\bar{x}, \tag{42}$$

where r_{xy} denotes the sample correlation coefficient and $\bar{x}$, $\bar{y}$ and s_x, s_y denote the arithmetic means and standard deviations of x and y, respectively.

t-test for single regression parameters:

$$t = \frac{b_i - \beta_i^*}{SE_{b_i}} \sim t(n-k-1) \quad \text{under} \quad H_0\colon \beta_i = \beta_i^*, \tag{43}$$

$$\text{where} \qquad SE_{b_{i-1}} = \sqrt{\text{MSE}(\mathbf{X}'\mathbf{X})^{-1}_{ii}} \tag{44}$$

$$\text{with} \qquad \text{MSE} = \frac{\text{SSE}}{n-(k+1)} = \frac{\sum(y_j - \hat{y}_j)^2}{n-(k+1)}. \tag{45}$$

In the special case of only 1 regressor:

$$SE_{b_1} = \sqrt{\frac{MSE}{SSX}} \quad \text{and} \quad SE_{b_0} = \sqrt{MSE\left(\frac{1}{n} + \frac{\bar{x}^2}{SSX}\right)}, \tag{46}$$

$$\text{where} \qquad SSX = \sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{47}$$

Confidence interval for regression parameters:

$$CI_{(1-\alpha)} = b_i \pm t_{\frac{\alpha}{2}}(n-k-1)SE_{b_i}. \tag{48}$$

Confidence interval on estimated mean at x:

$$CI_{(1-\alpha)} = \left[\hat{\mu}_{Y|\mathbf{x}} \pm t_{\frac{\alpha}{2}}(n-k-1)\sqrt{MSE \cdot \mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}}\,\right], \tag{49}$$

$$\text{where} \qquad \mathbf{x}' = (1, x_1, x_2, \ldots, x_k).$$

In the special case of only 1 regressor:

$$CI_{(1-\alpha)} = \hat{\mu}_y \pm t_{\alpha/2}(n-2) \cdot \sqrt{MSE\left(\frac{1}{n} + \frac{(x-\bar{x})^2}{SSX}\right)}. \tag{50}$$

Prediction interval on individual response at x:

$$CI_{(1-\alpha)} = \left[\hat{Y}|\mathbf{x} \pm t_{\frac{\alpha}{2}}(n-k-1)\sqrt{MSE \cdot (1 + \mathbf{x}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x})}\right], \tag{51}$$

$$\text{where} \qquad \mathbf{x}' = (1, x_1, x_2, \ldots, x_k).$$

In the special case of only 1 regressor:

$$CI_{(1-\alpha)} = \hat{y} \pm t_{\alpha/2}(n-2) \cdot \sqrt{MSE\left(1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{SSX}\right)}. \tag{52}$$

ANOVA Table for Multiple Regression

Source	Sum of Squares	DF	Mean Square	F Ratio
Regression	SSR=$\sum(\hat{y}_i - \bar{y})^2$	k	SSR/DFR	MSR/MSE
Error	SSE=$\sum(y_i - \hat{y}_i)^2$	$n-(k+1)$	SSE/DFE	
Total	SST=$\sum(y_i - \bar{y})^2$	$n-1$	SST/DFT	

where n is the number of observation tuples and k is the number of regressors (excluding the intercept).

The ANOVA F-test for Multiple Regression

The ratio $\frac{\text{MSR}}{\text{MSE}}$ is an F statistic for testing

$$H_0:\ \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

against

$$H_1:\ \text{Not all } \beta_i,\ i = 1, \ldots, k \text{ are zero.}$$

$$\text{Under } H_0: \qquad F = \frac{\text{MSR}}{\text{MSE}} = \frac{\text{SSR}/k}{\text{SSE}/(n-k-1)} \sim F(k, n-k-1) \tag{53}$$

$$\text{and} \quad F = \frac{R^2}{1-R^2} \cdot \frac{n-(k+1)}{k} \sim F(k, n-k-1). \tag{54}$$

The coefficient of determination R^2:

$$R^2 = r_{xy}^2 = \frac{s_{\hat{y}}^2}{s_y^2} = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}. \tag{55}$$

The adjusted coefficient of determination $\overline{R^2}$:

$$\overline{R^2} = 1 - \frac{\text{MSE}}{\text{MST}} = 1 - (1-R^2)\frac{n-1}{n-(k+1)}. \tag{56}$$

The partial F-test for adding/deleting regressors

Consider the full regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k + \epsilon$$

with sum of squared residuals SSE_F and mean square error MSE_F as an alternative to the reduced model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_{k-r} X_{k-r} + \epsilon$$

with sum of squared residuals SSE_R. Then

$$F = \frac{(SSE_R - SSE_F)/r}{MSE_F} \sim F(r, n-(k+1)) \qquad (57)$$

$$\text{under } H_0: \quad \beta_{k-r+1} = \beta_{k-r+2} = \ldots = \beta_k = 0.$$

The Durbin-Watson test

In order to test for first order autocorrelation ρ_1 at significance level α:

1. Choose a significance level (e.g. $\alpha = 0.05$).
2. Calculate $d = \dfrac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2} \approx 2(1 - \hat{\rho}_1)$.
3. Look up:
   $d_L(\frac{\alpha}{2})$ and $d_U(\frac{\alpha}{2})$ for a two-sided test,
   $d_L(\alpha)$ and $d_U(\alpha)$ for a one-sided test.
4. (i) Two-sided: $H_0: \rho_1 = 0$ vs. $H_1: \rho_1 \neq 0$
   - $d \leq d_L$ or $d \geq 4 - d_L$ $\Rightarrow$ reject H_0.
   - $d_U \leq d \leq 4 - d_U$ $\Rightarrow$ accept H_0.
   - otherwise $\Rightarrow$ inconclusive.

   (ii) One-sided: $H_0: \rho_1 = 0$ vs. $H_1: \rho_1 > 0$
   - $d \leq d_L$ $\Rightarrow$ reject H_0.
   - $d \geq d_U$ $\Rightarrow$ accept H_0.
   - otherwise $\Rightarrow$ inconclusive.

   (iii) One-sided: $H_0: \rho_1 = 0$ vs. $H_1: \rho_1 < 0$
   - $d \geq 4 - d_L$ $\Rightarrow$ reject H_0.
   - $d \leq 4 - d_U$ $\Rightarrow$ accept H_0.
   - otherwise $\Rightarrow$ inconclusive.

Multicollinearity

The variance inflation factor associated with the regressor X_i is

$$\text{VIF}(b_i) = \frac{1}{1 - R_i^2}, \qquad (58)$$

where R_i^2 denotes the coefficient of determination in a regression of X_i on all other regressors.

Logistic Regression

The binary logistic regression model is

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x, \tag{59}$$

where p is the probability of success and x is the explanatory variable.

The probability of success is then given by the logistic function

$$p = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}}. \tag{60}$$

Confidence Intervals for logistic regression:

A level α confidence interval for β_1 (slope) is

$$CI_{(1-\alpha)} = b_1 \pm z_{\alpha/2} SE_{b_1}. \tag{61}$$

A level α confidence interval for the odds ratio e^{β_1} is

$$CI_{(1-\alpha)} = \left[e^{b_1 - z_{\alpha/2} SE_{b_1}}, e^{b_1 + z_{\alpha/2} SE_{b_1}}\right]. \tag{62}$$

Significance tests for logistic regression:

$$z = \frac{b_1}{SE_{b_1}} \sim N(0,1) \quad \text{and} \quad z^2 = \left(\frac{b_1}{SE_{b_1}}\right)^2 \sim \chi^2(1) \quad \text{under } H_0\colon \beta_1 = 0. \tag{63}$$

Wilcoxon Signed-Rank Test:

The table below displays the integer numbers which correspond most closely to the critical values t_p of rejecting equal medians in one-sided Wilcoxon Signed-Rank tests at significance level p. n denotes the number of observation pairs.

p :	0.05	0.025	0.01	0.005
$2p$:	0.10	0.05	0.02	0.01
$n = 6$	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3
11	14	11	7	5
12	17	14	10	7
13	21	17	13	10
14	26	21	16	13
15	30	25	20	16
16	36	30	24	19
17	41	35	28	23
18	47	40	33	28
19	54	46	38	32
20	60	52	43	37

Critical Values for the Durbin-Watson Statistic (d): n = sample size, k = number of regressors (excluding intercept)

		k = 1:		k = 2:		k = 3:		k = 4:		k = 5:	
n	alpha	dL	dU	dL	dU	dL	dU	dL	dU	dL	dU
	0.01	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.97
15	0.025	0.95	1.22	0.83	1.40	0.71	1.61	0.59	1.85	0.48	2.10
	0.05	1.08	1.36	0.95	1.54	0.82	1.75	0.69	1.97	0.56	2.21
	0.01	0.95	1.15	0.86	1.27	0.77	1.41	0.68	1.57	0.60	1.74
20	0.025	1.08	1.29	0.99	1.41	0.89	1.55	0.79	1.71	0.70	1.87
	0.05	1.20	1.41	1.10	1.54	1.00	1.68	0.90	1.83	0.79	1.99
	0.01	1.05	1.21	0.98	1.30	0.91	1.41	0.83	1.52	0.76	1.65
25	0.025	1.18	1.34	1.10	1.43	1.02	1.54	0.94	1.65	0.86	1.77
	0.05	1.29	1.45	1.21	1.55	1.12	1.66	1.04	1.77	0.95	1.89
	0.01	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
30	0.025	1.25	1.38	1.18	1.46	1.12	1.54	1.05	1.63	0.98	1.73
	0.05	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
	0.01	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
40	0.025	1.35	1.45	1.30	1.51	1.25	1.56	1.20	1.63	1.14	1.69
	0.05	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
	0.01	1.32	1.40	1.29	1.45	1.25	1.49	1.21	1.54	1.16	1.59
50	0.025	1.42	1.50	1.38	1.54	1.34	1.59	1.30	1.64	1.25	1.69
	0.05	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
	0.01	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
60	0.025	1.47	1.54	1.44	1.57	1.40	1.61	1.37	1.65	1.33	1.69
	0.05	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
	0.01	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
80	0.025	1.54	1.59	1.52	1.62	1.49	1.65	1.47	1.67	1.44	1.70
	0.05	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
	0.01	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65
100	0.025	1.59	1.63	1.57	1.65	1.55	1.67	1.53	1.70	1.51	1.72
	0.05	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

Critical Values q(k,N-k) of the Studentized Range Distribution for alpha=0.05:

N-k: \ k:	2	3	4	5	6	7	8	9	10
1	17.96	26.96	32.81	37.06	40.39	43.10	45.39	47.37	49.09
2	6.08	8.33	9.80	10.88	11.73	12.43	13.03	13.54	13.99
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15
17	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92
30	2.89	3.49	3.85	4.10	4.30	4.46	4.60	4.72	4.82
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.73
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65
120	2.80	3.36	3.68	3.92	4.10	4.24	4.36	4.47	4.56
Infinity	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47

Critical Values q(k,N-k) of the Studentized Range Distribution for alpha=0.01:

N-k: \ k:	2	3	4	5	6	7	8	9	10
1	89.99	134.77	163.93	185.22	201.80	215.28	226.60	236.34	244.88
2	14.04	19.02	22.29	24.72	26.63	28.20	29.53	30.68	31.69
3	8.26	10.62	12.17	13.32	14.24	15.00	15.64	16.20	16.69
4	6.51	8.12	9.17	9.96	10.58	11.10	11.54	11.93	12.26
5	5.70	6.98	7.80	8.42	8.91	9.32	9.67	9.97	10.24
6	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10
7	4.95	5.92	6.54	7.00	7.37	7.68	7.94	8.17	8.37
8	4.75	5.64	6.20	6.62	6.96	7.24	7.47	7.68	7.86
9	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.33	7.49
10	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21
11	4.39	5.15	5.62	5.97	6.25	6.48	6.67	6.84	6.99
12	4.32	5.05	5.50	5.84	6.10	6.32	6.51	6.67	6.81
13	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67
14	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54
15	4.17	4.84	5.25	5.56	5.80	5.99	6.16	6.31	6.44
16	4.13	4.79	5.19	5.49	5.72	5.92	6.08	6.22	6.35
17	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27
18	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20
19	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14
20	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09
24	3.96	4.55	4.91	5.17	5.37	5.54	5.69	5.81	5.92
30	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76
40	3.82	4.37	4.70	4.93	5.11	5.26	5.39	5.50	5.60
60	3.76	4.28	4.59	4.82	4.99	5.13	5.25	5.36	5.45
120	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30
Infinity	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16

The Standard Normal Distribution Φ

Example. $P(Z \leq 1.34) = \Phi(1.34) = 0.9099.$

$\Phi(z) = P(Z \leq z)$

Second decimal of z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Φ is usually tabulated only for positive values of z, because due to the symmetry of ϕ around zero:

$$\Phi(-z) = P(Z \leq -z) = P(Z > z) = 1 - P(Z \leq z) = 1 - \Phi(z).$$

Example. $Z \sim N(0, 1)$, then:

$$P(Z \leq -1) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.$$

Table. Tail fractiles (t_α) of the t-distribution: $P(T > t_\alpha(df)) = \alpha$.

df	t_α 0.100	0.050	0.025	0.010	0.005	0.001	0.0005
1	3.078	6.314	12.706	31.821	63.657	318.309	636.619
2	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	1.299	1.676	2.009	2.403	2.678	3.261	3.496
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
70	1.294	1.667	1.994	2.381	2.648	3.211	3.435
80	1.292	1.664	1.990	2.374	2.639	3.195	3.416
90	1.291	1.662	1.987	2.368	2.632	3.183	3.402
100	1.290	1.660	1.984	2.364	2.626	3.174	3.390
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

The symmetry of the Student t-distribution implies

$$P(T > t_\alpha(df)) = \alpha \;\Leftrightarrow\; P(|T| > t_\alpha(df)) = 2\alpha.$$

Example. $T \sim t(10) \Rightarrow\; 0.025 = P(T \geq 2.228),$

$$0.05 = P(|T| \geq 2.228).$$

Table. Tail fractiles χ^2_α of the χ^2-distribution: $P(\chi^2 > \chi^2_\alpha(df)) = \alpha$.

					χ_α^2(df)					
df / α	0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.001
1	0.000039	0.000157	0.000982	0.003932	0.0158	2.706	3.841	5.024	6.635	10.827
2	0.0100	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210	13.815
3	0.0717	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	16.266
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	18.466
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086	20.515
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	22.457
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	24.321
8	1.344	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090	26.124
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	27.877
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	29.588
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	31.264
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	32.909
13	3.565	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688	34.527
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	36.124
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	37.698
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	39.252
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	40.791
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	42.312
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	43.819
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	45.314
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	46.796
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	48.268
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	49.728
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	51.179
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	52.619
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	54.051
27	11.808	12.878	14.573	16.151	18.114	36.741	40.113	43.195	46.963	55.475
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	56.892
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	58.301
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	59.702
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	73.403
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	86.660
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	99.608
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	112.317
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	124.839
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	137.208
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	149.449

Example:

$$\chi^2 \sim \chi^2(16) \Rightarrow \; 0.05 = P(\chi^2 \geq 26.296).$$

$F_{0.1}(m,n)$

n \m	1	2	3	4	5	6	7	8	9	10	15	30	40	60	120
1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	61.22	62.26	62.53	62.79	63.06
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.42	9.46	9.47	9.47	9.48
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.20	5.17	5.16	5.15	5.14
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.87	3.82	3.80	3.79	3.78
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.24	3.17	3.16	3.14	3.12
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.87	2.80	2.78	2.76	2.74
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.63	2.56	2.54	2.51	2.49
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.46	2.38	2.36	2.34	2.32
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.34	2.25	2.23	2.21	2.18
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.24	2.16	2.13	2.11	2.08
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.17	2.08	2.05	2.03	2.00
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.10	2.01	1.99	1.96	1.93
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.05	1.96	1.93	1.90	1.88
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.01	1.91	1.89	1.86	1.83
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	1.97	1.87	1.85	1.82	1.79
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.94	1.84	1.81	1.78	1.75
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.91	1.81	1.78	1.75	1.72
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.89	1.78	1.75	1.72	1.69
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.86	1.76	1.73	1.70	1.67
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.84	1.74	1.71	1.68	1.64
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.83	1.72	1.69	1.66	1.62
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.81	1.70	1.67	1.64	1.60
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.80	1.69	1.66	1.62	1.59
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.78	1.67	1.64	1.61	1.57
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.77	1.66	1.63	1.59	1.56
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.76	1.65	1.61	1.58	1.54
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.75	1.64	1.60	1.57	1.53
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.74	1.63	1.59	1.56	1.52
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.73	1.62	1.58	1.55	1.51
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.72	1.61	1.57	1.54	1.50
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.66	1.54	1.51	1.47	1.42
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.60	1.48	1.44	1.40	1.35
80	2.77	2.37	2.15	2.02	1.92	1.85	1.79	1.75	1.71	1.68	1.57	1.44	1.40	1.36	1.31
100	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69	1.66	1.56	1.42	1.38	1.34	1.28
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.49	1.34	1.30	1.24	1.17

$F_{0.05}(m,n)$

n \m	1	2	3	4	5	6	7	8	9	10	15	30	40	60	120
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	245.95	250.10	251.14	252.20	253.25
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.43	19.46	19.47	19.48	19.49
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.70	8.62	8.59	8.57	8.55
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.86	5.75	5.72	5.69	5.66
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.62	4.50	4.46	4.43	4.40
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.94	3.81	3.77	3.74	3.70
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.51	3.38	3.34	3.30	3.27
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.22	3.08	3.04	3.01	2.97
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.01	2.86	2.83	2.79	2.75
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.85	2.70	2.66	2.62	2.58
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.72	2.57	2.53	2.49	2.45
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.62	2.47	2.43	2.38	2.34
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.53	2.38	2.34	2.30	2.25
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.46	2.31	2.27	2.22	2.18
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.40	2.25	2.20	2.16	2.11
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.35	2.19	2.15	2.11	2.06
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.31	2.15	2.10	2.06	2.01
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.27	2.11	2.06	2.02	1.97
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.23	2.07	2.03	1.98	1.93
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.20	2.04	1.99	1.95	1.90
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.18	2.01	1.96	1.92	1.87
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.15	1.98	1.94	1.89	1.84
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.13	1.96	1.91	1.86	1.81
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.11	1.94	1.89	1.84	1.79
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.09	1.92	1.87	1.82	1.77
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.07	1.90	1.85	1.80	1.75
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.06	1.88	1.84	1.79	1.73
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.04	1.87	1.82	1.77	1.71
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.03	1.85	1.81	1.75	1.70
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.01	1.84	1.79	1.74	1.68
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	1.92	1.74	1.69	1.64	1.58
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.84	1.65	1.59	1.53	1.47
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.79	1.60	1.54	1.48	1.41
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.77	1.57	1.52	1.45	1.38
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.67	1.46	1.39	1.32	1.22

$F_{0.025}(m,n)$

n \m	1	2	3	4	5	6	7	8	9	10	15	30	40	60	120
1	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28	968.63	984.87	1001.41	1005.60	1009.80	1014.02
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.43	39.46	39.47	39.48	39.49
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.25	14.08	14.04	13.99	13.95
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.66	8.46	8.41	8.36	8.31
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.43	6.23	6.18	6.12	6.07
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.27	5.07	5.01	4.96	4.90
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.57	4.36	4.31	4.25	4.20
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.10	3.89	3.84	3.78	3.73
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.77	3.56	3.51	3.45	3.39
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.52	3.31	3.26	3.20	3.14
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.33	3.12	3.06	3.00	2.94
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.18	2.96	2.91	2.85	2.79
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.05	2.84	2.78	2.72	2.66
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	2.95	2.73	2.67	2.61	2.55
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.86	2.64	2.59	2.52	2.46
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.79	2.57	2.51	2.45	2.38
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.72	2.50	2.44	2.38	2.32
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.67	2.44	2.38	2.32	2.26
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.62	2.39	2.33	2.27	2.20
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.57	2.35	2.29	2.22	2.16
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.53	2.31	2.25	2.18	2.11
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.50	2.27	2.21	2.14	2.08
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.47	2.24	2.18	2.11	2.04
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.44	2.21	2.15	2.08	2.01
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.41	2.18	2.12	2.05	1.98
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.39	2.16	2.09	2.03	1.95
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.36	2.13	2.07	2.00	1.93
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.34	2.11	2.05	1.98	1.91
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.32	2.09	2.03	1.96	1.89
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.31	2.07	2.01	1.94	1.87
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.18	1.94	1.88	1.80	1.72
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.06	1.82	1.74	1.67	1.58
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28	2.21	2.00	1.75	1.68	1.60	1.51
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	1.97	1.71	1.64	1.56	1.46
∞	5.02	3.69	3.13	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.83	1.57	1.48	1.39	1.27

$F_{0.01}(m,n)$

n \m	1	2	3	4	5	6	7	8	9	10	15	30	40	60	120
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6157	6261	6287	6313	6339
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	26.9	26.5	26.4	26.3	26.2
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.2	13.8	13.7	13.7	13.6
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.72	9.38	9.29	9.20	9.11
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.56	7.23	7.14	7.06	6.97
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.31	5.99	5.91	5.82	5.74
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.52	5.20	5.12	5.03	4.95
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	4.96	4.65	4.57	4.48	4.40
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.56	4.25	4.17	4.08	4.00
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.25	3.94	3.86	3.78	3.69
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.01	3.70	3.62	3.54	3.45
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.82	3.51	3.43	3.34	3.25
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.66	3.35	3.27	3.18	3.09
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.52	3.21	3.13	3.05	2.96
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.41	3.10	3.02	2.93	2.84
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.31	3.00	2.92	2.83	2.75
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.23	2.92	2.84	2.75	2.66
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.15	2.84	2.76	2.67	2.58
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.09	2.78	2.69	2.61	2.52
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.03	2.72	2.64	2.55	2.46
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	2.98	2.67	2.58	2.50	2.40
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	2.93	2.62	2.54	2.45	2.35
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	2.89	2.58	2.49	2.40	2.31
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.85	2.54	2.45	2.36	2.27
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.81	2.50	2.42	2.33	2.23
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.78	2.47	2.38	2.29	2.20
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.75	2.44	2.35	2.26	2.17
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.73	2.41	2.33	2.23	2.14
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.70	2.39	2.30	2.21	2.11
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.52	2.20	2.11	2.02	1.92
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.35	2.03	1.94	1.84	1.73
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.27	1.94	1.85	1.75	1.63
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.22	1.89	1.80	1.69	1.57
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.04	1.70	1.59	1.47	1.32