Generalized ARCH models

In practice the ARCH needs fairly many lags. Usually far less lags are needed by modifying the model to

$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1},$$

with $\omega > 0$, $\alpha > 0$, $\delta \geq 0$, and $\alpha + \delta < 1$. The model is called the Generalized ARCH (GARCH) model. Usually the above GARCH(1,1) is adequate in practice.

Econometric packages call α (coefficient of u_{t-1}^2) the ARCH parameter and δ (coefficient of h_{t-1}) the GARCH parameter.

Note again that defining $\nu_t = u_t^2 - h_t$, we get

$$u_t^2 = \omega + (\alpha + \delta)u_{t-1}^2 + \nu_t - \delta\nu_{t-1},$$

a heteroscedastic ARMA(1,1) process.

Applying backward substitution, we get

$$\begin{aligned}
h_t =& \omega + \alpha u_{t-1}^2 + \delta h_{t-1} \\
=& \omega + \alpha u_{t-1}^2 + \delta(\omega + \alpha u_{t-2}^2 + \delta h_{t-2}) \\
=& \omega(1+\delta) + \alpha(u_{t-1}^2 + \delta u_{t-2}^2) + \delta^2 h_{t-2} \\
&\vdots \\
=& \omega \sum_{i=0}^{n-1} \delta^i + \alpha \sum_{j=0}^{n-1} \delta^{j-1} u_{t-j}^2 + \delta^n h_{t-n} \\
\overset{n\to\infty}{\longrightarrow}& \ \frac{\omega}{1-\delta} + \alpha \sum_{j=1}^{\infty} \delta^{j-1} u_{t-j}^2
\end{aligned}$$

an ARCH(∞) process. Thus the GARCH term captures all the history from $t-2$ backwards of the shocks u_t.

Imposing additional lag terms, the model can be extended to GARCH(p, q) model

$$h_t = \omega + \sum_{i=1}^{q} \alpha u_{t-i}^2 + \sum_{j=1}^{p} \delta_j h_{t-j}$$

[c.f. ARMA(p, q)]. Nevertheless, as noted above, in practice GARCH(1,1) is adequate.

Example. MA(1)-GARCH(1,1) model of Nasdaq returns. The model is

$$r_t = \mu + u_t + \theta u_{t-1}$$
$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1}.$$

Estimation results (EViews 4.0)

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 21 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.084907	0.017700	4.797124	0.0000
MA(1)	0.171620	0.020952	8.190983	0.0000
	Variance Equation			
C	0.027892	0.009213	3.027258	0.0000
ARCH(1)	0.121770	0.020448	5.955103	0.0000
GARCH(1)	0.857095	0.021526	39.81666	0.0000

R-squared	0.007104	Mean dependent var	0.086119
Adjusted R-squared	0.005685	S.D. dependent var	1.097336
S.E. of regression	1.094213	Akaike info criterion	2.695856
Sum squared resid	3352.444	Schwarz criterion	2.706443
Log likelihood	-3775.938	F-statistic	5.008069
Durbin-Watson stat	2.129878	Prob(F-statistic)	0.000507
Inverted MA Roots	-.17		

Correlogram of Standardized Residuals Squared

Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.004	0.004	0.0519	
		2	0.035	0.035	3.5358	0.060
		3	-0.007	-0.007	3.6727	0.159
		4	-0.007	-0.008	3.8082	0.283
		5	-0.009	-0.008	4.0159	0.404
		6	0.001	0.001	4.0179	0.547
		7	-0.021	-0.020	5.2184	0.516
		8	-0.023	-0.023	6.7031	0.460
		9	-0.019	-0.017	7.7149	0.462
		10	-0.016	-0.014	8.4002	0.494
		11	-0.024	-0.023	9.9573	0.444
		12	-0.008	-0.008	10.148	0.517
		13	-0.007	-0.006	10.278	0.592
		14	-0.004	-0.005	10.324	0.667
		15	0.005	0.004	10.405	0.732
		16	-0.004	-0.005	10.448	0.791
		17	-0.008	-0.010	10.639	0.831
		18	-0.025	-0.027	12.405	0.775
		19	0.002	0.001	12.421	0.825
		20	-0.030	-0.030	14.903	0.729
		21	0.000	-0.002	14.903	0.782
		22	-0.016	-0.016	15.671	0.788
		23	-0.004	-0.005	15.710	0.830
		24	0.030	0.030	18.231	0.745
		25	-0.012	-0.014	18.627	0.772
		26	-0.012	-0.016	19.046	0.795
		27	0.011	0.010	19.387	0.820
		28	-0.015	-0.017	20.051	0.829
		29	0.021	0.018	21.360	0.810
		30	-0.002	-0.004	21.372	0.845
		31	-0.001	-0.004	21.377	0.876
		32	-0.008	-0.008	21.542	0.897
		33	0.037	0.036	25.370	0.791
		34	-0.016	-0.017	26.108	0.797
		35	0.009	0.007	26.352	0.823
		36	-0.013	-0.013	26.810	0.838

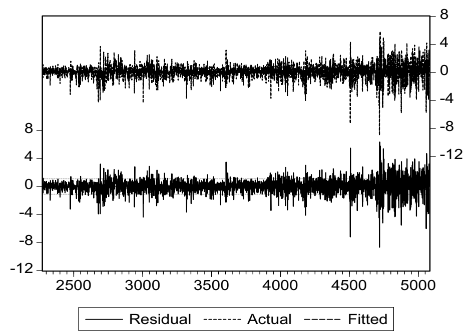


Figure. Conditional standard deviation function

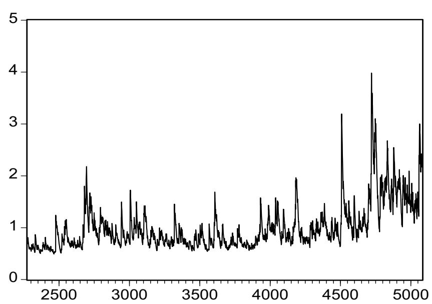
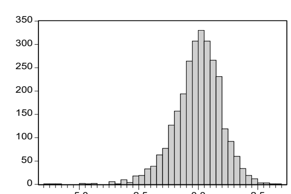


Figure. Conditional standard deviation function



Series: Standardized Residuals
Sample 2276 5080
Observations 2805

Mean	-0.019280
Median	0.050222
Maximum	3.265924
Minimum	-6.466048
Std. Dev.	0.999654
Skewness	-0.731417
Kurtosis	5.456912
Jarque-Bera	955.6050
Probability	0.000000

Figure. Conditional standard deviation function

The autocorrelations of the squared standardized residuals pass the white noise test. Nevertheless, the normality of the standardized residuals is strongly rejected. This is why robust standard errors are used in the estimation of the standard errors.

The variance function can be extended by including regressors (exogenous or predetermined variables), x_t, in it

$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1} + \pi x_t.$$

Note that if x_t can assume negative values, it may be desirable to introduce absolute values $|x_t|$ in place of x_t in the conditional variance function. For example with daily data a Monday dummy could be introduced in the model to capture the weekend non-trading in the volatility.

ARCH-M Model

The regression equation may be extended by introducing the variance function into the equation

$$y_t = \mathbf{x}_t'\beta + \gamma g(h_t) + u_t,$$

where $u_t \sim$ GARCH, and g is a suitable function (usually square root or logarithm).

This is called the ARCH in Mean (ARCH-M) model (Engle, Lilien and Robbins (1987)[¶]). The ARCH-M model is often used in finance where the expected return on an asset is related to the expected asset risk. The coefficient γ reflects the risk-return tradeoff.

[¶] *Econometrica*, 55, 391–407.

Example. Does the daily mean return of Nasdaq depend on the volatility level?

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 22 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.198064	0.074141	2.671456	0.0076
C	-0.069416	0.061432	-1.129969	0.2585
MA(1)	0.174785	0.020644	8.466806	0.0000
		Variance Equation		
C	0.031799	0.009301	3.419007	0.0006
ARCH(1)	0.134070	0.020974	6.392287	0.0000
GARCH(1)	0.842134	0.021350	39.44407	0.0000

R-squared	0.011379	Mean dependent var	0.086119
Adjusted R-squared	0.009613	S.D. dependent var	1.097336
S.E. of regression	1.092049	Akaike info criterion	2.694709
Sum squared resid	3338.007	Schwarz criterion	2.707413
Log likelihood	-3773.330	F-statistic	6.443432
Durbin-Watson stat	2.127550	Prob(F-statistic)	0.000006
Inverted MA Roots	-.17		

The volatility term in the mean equation is statistically significant indicating that rather than being constant the mean return is dependent on the level of volatility.

Consequently the data suggests that the best fitting model so far is of the form

$$r_t = \gamma\sqrt{h_t} + u_{t-1} + \theta u_{t-1}$$
$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1}.$$

Below are the estimation results for the above model

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 16 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.119204	0.022944	5.195479	0.0000
MA(1)	0.174104	0.020771	8.382098	0.0000
	Variance Equation			
C	0.031291	0.009545	3.278211	0.0010
ARCH(1)	0.133713	0.021011	6.363810	0.0000
GARCH(1)	0.843131	0.021785	38.70279	0.0000

R-squared	0.010861	Mean dependent var	0.086119
Adjusted R-squared	0.009448	S.D. dependent var	1.097336
S.E. of regression	1.092141	Akaike info criterion	2.694578
Sum squared resid	3339.759	Schwarz criterion	2.705165
Log likelihood	-3774.146	Durbin-Watson stat	2.132844
Inverted MA Roots	-.17		

Correlogram of Standardized Residuals Squared

Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.003	-0.003	0.0213	
		2	0.034	0.034	3.2371	0.072
		3	-0.008	-0.008	3.4198	0.181
		4	-0.009	-0.010	3.6300	0.304
		5	-0.009	-0.009	3.8611	0.425
		6	0.001	0.001	3.8636	0.569
		7	-0.022	-0.021	5.1942	0.519
		8	-0.023	-0.023	6.6228	0.469
		9	-0.018	-0.017	7.5391	0.480
		10	-0.015	-0.014	8.1998	0.514
		11	-0.023	-0.023	9.7327	0.464
		12	-0.007	-0.007	9.8585	0.543
		13	-0.006	-0.005	9.9568	0.620
		14	-0.001	-0.002	9.9594	0.697
		15	0.005	0.003	10.023	0.761
		16	-0.003	-0.005	10.055	0.816
		17	-0.006	-0.008	10.163	0.858
		18	-0.025	-0.027	11.907	0.806
		19	0.006	0.004	11.994	0.848
		20	-0.030	-0.030	14.490	0.754
		21	0.003	0.000	14.511	0.804
		22	-0.016	-0.015	15.193	0.813
		23	-0.002	-0.004	15.207	0.853
		24	0.031	0.031	17.937	0.761
		25	-0.011	-0.013	18.265	0.790
		26	-0.011	-0.015	18.602	0.816
		27	0.013	0.011	19.073	0.833
		28	-0.015	-0.016	19.737	0.842
		29	0.022	0.019	21.147	0.819
		30	-0.003	-0.004	21.173	0.853
		31	0.000	-0.002	21.173	0.882
		32	-0.006	-0.006	21.293	0.904
		33	0.041	0.040	26.022	0.763
		34	-0.016	-0.016	26.794	0.769
		35	0.009	0.006	27.013	0.797
		36	-0.013	-0.012	27.472	0.814

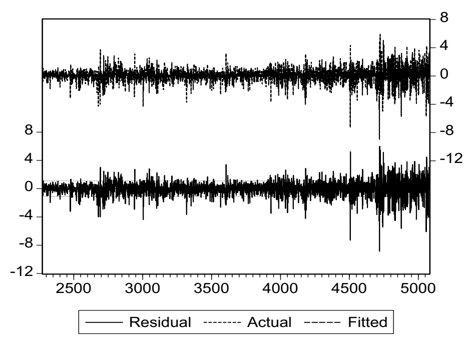
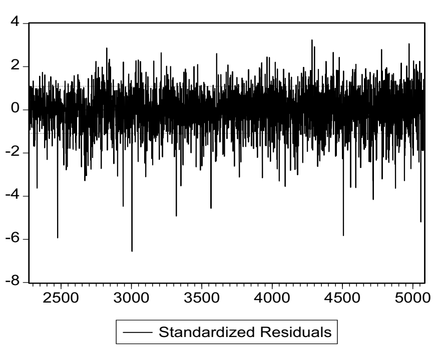
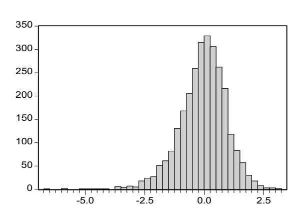


Figure. Actual and fitted series, and residuals





Series: Standardized Residuals	
Sample 2276 5080	
Observations 2805	
Mean	-0.056142
Median	0.018376
Maximum	3.239140
Minimum	-6.557423
Std. Dev.	0.998195
Skewness	-0.726040
Kurtosis	5.409519
Jarque-Bera	924.9855
Probability	0.000000

Looking at the standardized residuals, the distribution and the sample statistics of the distribution, we observe that the residual distribution is obviously skewed in addition to the leptokurtosis. The skewness may be due to some asymmetry in the conditional volatility which we have not yet modeled. In financial data the asymmetry is usually, such that downward shocks cause higher volatility in the near future than the positive shocks. In finance this is called the leverage effect.

An obvious and simple first hand check for the asymmetry is to investigate the cross autocorrelations between standardized and squared standardized GARCH residuals.

Below are the cross autocorrelations between the standardized and squared standardized residuals of the fitted MA(1)-GARCH(1,1) model.

```
============================================================
    Z,Z2(-i)              Z,Z2(+i)       i     lag    lead
============================================================
    ****|         |      ****|         | 0  -0.3916 -0.3916
        |         |         *|         | 1   0.0342 -0.0782
        |         |         *|         | 2  -0.0055 -0.0842
        |         |          |         | 3   0.0093 -0.0373
        |         |          |         | 4   0.0066  0.0315
        |         |          |         | 5   0.0134 -0.0046
        |         |          |         | 6  -0.0134 -0.0019
        |         |          |         | 7   0.0113 -0.0004
        |         |          |         | 8  -0.0019  0.0045
        |         |          |         | 9  -0.0034  0.0272
        |         |          |         | 10 -0.0205  0.0128
============================================================
```

The cross autocorrelations correlations are not large, but may indicate some asymmetry present.

Asymmetric ARCH: TARCH and EGARCH

A kind of stylized fact in stock markets is that downward movements are followed by higher volatility. EViews includes two models that allow for asymmetric shocks to volatility.

The TARCH model

Threshold ARCH, TARCH (Zakoian 1994, *Journal of Economic Dynamics and Control*, 931–955 , Glosten, Jagannathan and Runkle 1993, *Journal of Finance*, 1779-1801) is given by [TARCH(1,1)]

$$h_t = \omega + \alpha u_{t-1}^2 + \phi u_{t-1}^2 d_{t-1} + \delta h_{t-1},$$

where $d_t = 1$, if $u_t < 0$ (bad news) and zero otherwise. Thus the impact of good news is α while for the bad news $(\alpha + \phi)$. Hence, $\phi \neq 0$ implies asymmetry. The leverage exists if $\phi > 0$.

Example. Estimation results for the MA(1)-TARCH-M model.

```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 26 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
==============================================================
                 Coefficient  Std. Error  z-Statistic  Prob.
==============================================================
SQR(GARCH)         0.091184    0.023097    3.947880   0.0001
MA(1)              0.184263    0.020899    8.816678   0.0000
==============================================================
               Variance Equation
==============================================================
C                  0.037068     0.009513    3.896566   0.0001
ARCH(1)            0.084275     0.025080    3.360240   0.0008
(RESID<0)*ARCH(1)  0.099893     0.040881    2.443502   0.0145
GARCH(1)           0.833239     0.019001    43.85202   0.0000
==============================================================
R-squared          0.009604    Mean dependent var    0.086119
Adjusted R-squared 0.007835    S.D. dependent var    1.097336
S.E. of regression 1.093029    Akaike info criterion 2.686832
Sum squared resid  3344.000    Schwarz criterion     2.699536
Log likelihood    -3762.281    Durbin-Watson stat    2.149776
==============================================================
Inverted MA Roots        -.18
==============================================================
```

The goodness of fit improve, and the statistically significant positive asymmetry parameter indicates presence of leverage.

Furthermore, as seen below, the first few cross autocorrelations reduce to about one half of the original ones. They are still statistically significant, slightly exceeding the approximate 95% boundaries $\pm 2/\sqrt{T} = \pm 2/\sqrt{2805} \approx \pm 0.038$.

Cross autocorrelations of the standardized and squared standardized MA(1)-TARCH(1,1)-M model.

Z,Z2(-i)	Z,Z2(+i)	i	lag	lead
****| |	****| |	0	-0.3543	-0.3543
| |	*| |	1	0.0304	-0.0488
| |	*| |	2	-0.0071	-0.0537
| |	| |	3	0.0112	-0.0156
| |	|* |	4	0.0069	0.0545
| |	| |	5	0.0113	0.0080

The EGARCH model

Nelson (1991) (*Econometrica*, 347–370) proposed the Exponential GARCH (EGARCH) model for the variance function of the form (EGARCH(1,1))

$$\log h_t = \omega + \delta \log h_{t-1} + \alpha\,|z_{t-1}| + \phi z_{t-1},$$

where $z_t = u_t/\sqrt{h_t}$ is the standardized shock. Again the impact is asymmetric if $\phi \neq 0$, but here leverage is present if $\phi < 0$.

<u>Example</u> MA(1)-EGARCH(1,1)-M estimation results.

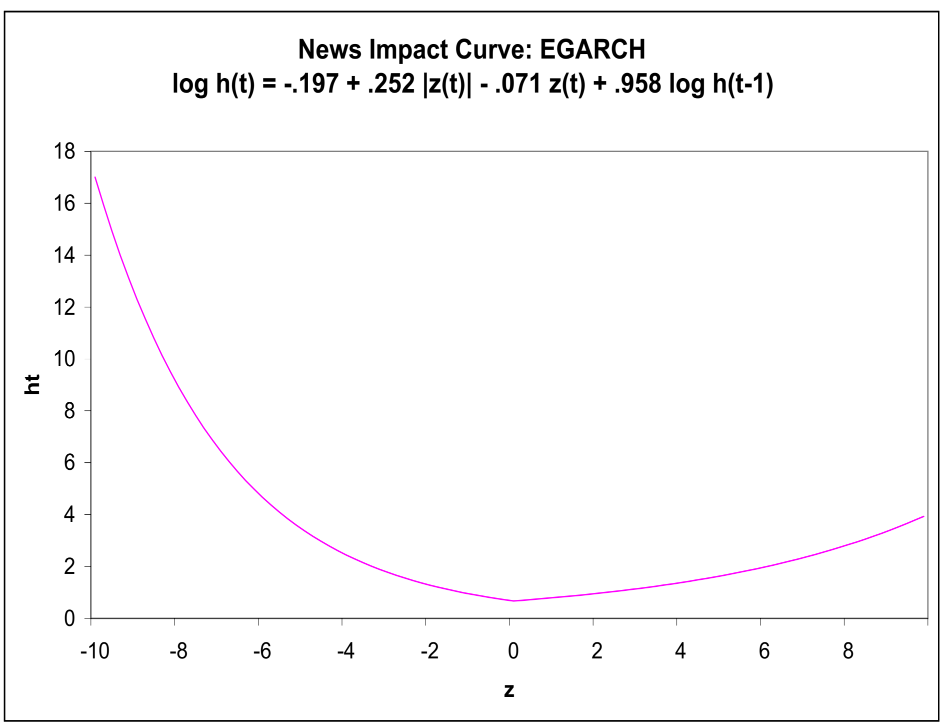
```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 28 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
==========================================================
                  Coefficient Std. Error z-Stat    Prob.
==========================================================
SQR(GARCH)           0.084631  0.022593  3.745866  0.0002
MA(1)                0.171543  0.020387  8.414399  0.0000
==========================================================
                       Variance Equation
==========================================================
C                   -0.197193  0.023051 -8.554804  0.0000
|RES|/SQR[GARCH](1)  0.251752  0.030816  8.169621  0.0000
RES/SQR[GARCH](1)   -0.071425  0.024034 -2.971755  0.0030
EGARCH(1)            0.958125  0.010941  87.57385  0.0000
==========================================================
R-squared          0.010762  Mean dependent var  0.086119
Adjusted R-squared 0.008995  S.D. dependent var  1.097336
S.E. of regression 1.092390  Akaike info criter  2.682518
Sum squared resid  3340.093  Schwarz criterion   2.695222
Log likelihood    -3756.232  Durbin-Watson stat  2.124928
==========================================================
Inverted MA Roots       -.17
==========================================================
```

Cross autocorrelations (not shown here) are about the same as with the TARCH model (i.e., disappear). Thus TARCH and EGARCH capture most part of the leverage effect.

News Impact Curve

The asymmetry of the conditional volatility function can be conveniently illustrated by the news impact curve (NIC). The curve is simply the graph of $h_t(z)$, where z indicates the shocks (news).

Below is a graph for the NIC of the above estimate EGARCH variance function, where h_{t-1} is replaced by the median of the estimated EGARCH series.



The Component ARCH Model

We can write the GARCH(1,1) model as

$$h_t = \bar{\omega} + \alpha(u_{t-1}^2 - \bar{\omega}) + \delta(h_{t-1} - \bar{\omega}),$$

where (exercise)

$$\bar{\omega} = \frac{\omega}{1 - \alpha - \delta}$$

is the unconditional variance of the series. Thus the usual GARCH has a mean reversion tendency towards $\bar{\omega}$. A further extension is to allow this unconditional or long term volatility to vary over time. This leads to the so called component ARCH that allows mean reversion to a varying level q_t instead of $\bar{\omega}$. The model is

$$\begin{aligned} h_t - q_t &= \alpha(u_{t-1}^2 - q_{t-1}) + \delta(h_{t-1} - q_{t-1}) \\ q_t &= \omega + \rho(q_{t-1} - \omega) + \theta(u_{t-1}^2 - h_{t-1}). \end{aligned}$$

An asymmetric version for the model is

$$\begin{aligned} h_t - q_t &= \alpha(u_{t-1}^2 - q_{t-1}) + \phi(u_{t-1}^2 - q_{t-1})d_{t-1} \\ &\quad + \delta(h_{t-1} - q_{t-1}) \\ q_t &= \omega + \rho(q_{t-1} - \omega) + \theta(u_{t-1}^2 - h_{t-1}). \end{aligned}$$

Example Asymmetric Component ARCH of the Nasdaq composite returns.

```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 4 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
==================================================================
                     Coefficient  Std. Error z-Statistic    Prob.
==================================================================
SQR(GARCH)              0.097235    0.023997   4.052002    0.0001
MA(1)                   0.182908    0.027822   6.574119    0.0000
==================================================================
                      Variance Equation
=================================================================
Perm: C                 0.926329    0.080794   11.46533    0.0000
Perm: [Q-C]             0.734067    0.077885   9.425047    0.0000
Perm: [ARCH-GARCH]      0.228360    0.048519   4.706581    0.0000
Tran: [ARCH-Q]          0.037121    0.039565   0.938228    0.3481
Tran: (RES<0)*[ARCH-Q] -0.077747    0.076100  -1.021635    0.3070
Tran: [GARCH-Q]        -0.688202    0.353158  -1.948710    0.0513
==================================================================
R-squared               0.008632  Mean dependent var     0.086119
Adjusted R-squared      0.006151  S.D. dependent var     1.097336
S.E. of regression      1.093956  Akaike info criterion  2.771137
Sum squared resid       3347.282  Schwarz criterion      2.788076
Log likelihood         -3878.519  Durbin-Watson stat     2.152828
==================================================================
Inverted MA Roots         -.18
==================================================================
```

This model, however, does not fit well into the data. Thus it seems that the best fitting models so far are either the TARCH or EGARCH.