A SYSTEM THEORETIC APPROACH TO THE THEORY OF DISCRETE FINITE - NODE MARKOV PROCESSES

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Abstract. In the paper the basic properties and behaviour of discrete finite-node (-state) Markov processes are dealt with. The processes to be considered are of general time-varying type: the transition probabilities may change from one transition to the next.

In the paper a system theoretic approach to the theory of Markov processes is introduced. It is shown that the basic concepts of general system theory have meaningful equivalents among the quantities of Markov processes. For the basic concepts of system theory the following form and purport will be derived.

1^o. Input is a sequence of transition probability matrices $P(t)$ over the observation period $[t_0, t_T]$: $u=[P(t_0), \ldots, P(t_{T-1})]$.

2^0. Output is a sequence of node probability vectors $\Pi(t)$ over the observation period $[t_0, t_T]$: $y=[\Pi(t_1), \ldots, \Pi(t_T)]$.

3^o. (a) Initial state of the system is the node probability vector at time t_0: $\Pi(t_0)=[\pi_1(t_0), \pi_2(t_0), \ldots, \pi_N(t_0)]$.

(b) State of the system at an arbitrary time $t_k \in [t_0, t_T]$ is shown to be the pair $(\Pi(t_0), u_{t_0,t_k})$, where u_{t_0,t_k} is the fragment $[P(t_0), \ldots, P(t_{k-1})]$ of the input u.

It is further shown that, with a fixed initial state Π^o, the input uniquely determines the output, i.e. mapping $\Gamma(\Pi^o)$: $\{u\} \rightarrow \{y\}$ is a function. Furthermore it is shown that the state of the system acts as the memory of the system: by knowing the state of the system and the future input into it, the future behaviour (output) of the system is determined. In the end, interpretations of the system presentation will be dealt with.

1. INTRODUCTION

This paper deals with basic properties and behaviour of a discrete finite-node Markov process. The topic is approached from the point of view of general system theory. At the same time the paper is a description about elementary system theory within the framework of an easily manageable mathematical model. The contents of the paper is based on the theory of discrete Markov processes presented in R.A. HOWARD´s book "Dynamic Probabilistic Systems, Volume I: Markov Models"[1]. Howard´s theory is now translated into the language of general system theory. In this translation the formalisms introduced in S. SALOVAARA´s[2] and P. MALASKA´s[3] papers are adapted. The considerations are carried out in rather a high level of abstraction, although interpretation of concepts and results, and possibilities to apply the theory are at least to some extent referred to.

2. DISCRETE FINITE-NODE MARKOV PROCESSES

2.1. Basic concepts

Let´s consider a certain, as yet more closely nonspecified phenomenon ("the real system"), which is observed or at least

1. R.A. Howard: Dynamic Probabilistic Systems, Volume I: Markov Models, John Wiley & Sons, Inc., New York 1971.

2. S. Salovaara: On set theoretical basic concepts of system theory, a discourse in a colloquium in University of Helsinki 14.3.1968 (in finnish).

3. P. Malaska: An economic model as an example of system theory, a discussion paper in a seminar in The Turku School of Economics 1973 (in finnish).

can be thought to become observed at discrete time points t_0, $t_1,\ldots,t_n,\ldots$ An exact description of the phenomenon at time t_n ($n=0,1,\ldots$) is usually called <u>the state</u> of the phenomenon at that time. The state can be expressed e.g. as a verbal description, as values of the variables relevant to considerations etc. In the following we assume that the phenomenon has only a finite number of possible states. These states are numbered $1,2,\ldots,N$.

When time goes from t_n to the next observation instant t_{n+1} the values of the state variables change in general, the state of the phenomenon changes. This kind of change is called a (state) <u>transition</u>. Further it is assumed that the phenomenon is stochastic by nature, whereupon the transitions are governed by certain probability laws, the <u>transition probabilities</u>. In a general case, the transition probabilities at a time t_n depend on both the time t_n and the states of the phenomenon at the times t_0, $t_1,\ldots,t_n$. When we denote by symbol X the quantity that shows the state of the phenomenon at any time t_n, i.e.

(1) $X(t_n)$: the state of the phenomenon at time t_n ($n=0,1,\ldots$) is $X(t_n)$, where $X(t_n)=1,2,\ldots,N$,

so we can state that X is a random variable, the distribution of which is determined on the basis of the transition probabilities. In the general case, the transition probabilities are conditional probabilities

(2) $$P\{X(t_{n+1})=j \mid X(t_n)=i_n, X(t_{n-1})=i_{n-1},\ldots,X(t_0)=i_0\},$$

where $n=0,1,\ldots$ and $1\le i_0,i_1,\ldots,i_n,j\le N$. The infinite-dimensional random variable

(3) $$X = [X(t_0),X(t_1),\ldots,X(t_n),\ldots]$$

with the transition probabilities given in (2), is called a stochastic process corresponding to the phenomenon in question. Any observed sequence of states, corresponding successive time points $t_0, t_1, \ldots, t_n, \ldots$,

(4) $$[x(t_0), x(t_1), \ldots, x(t_n), \ldots],$$

is a trajectory of this process.

From expression (2) we can see that the amount of information needed for managing the process is, in the general case, very large. Because of this considerations are often limited to Markov processes: it is assumed that the future behaviour of the process is only affected by the last state occupied by the process. Under this assumption it is

(5) $$P\{X(t_{n+1})=j \mid X(t_n)=i_n, X(t_{n-1})=i_{n-1}, \ldots, X(t_0)=i_0\} = P\{X(t_{n+1})=j \mid X(t_n)=i_n\}.$$

So the transition probabilities of a Markov process are of the form

(6) $$P\{X(t_{n+1})=j \mid X(t_n)=i\} = p_{ij}(t_n), \quad 1 \le i,j \le N, \quad n=0,1,\ldots$$

The transition probabilities can also be shown as a matrix

(7) $$P(t_n) = \{p_{ij}(t_n)\} = \begin{bmatrix} p_{11}(t_n) & p_{12}(t_n) & \ldots & p_{1N}(t_n) \\ p_{21}(t_n) & p_{22}(t_n) & \ldots & p_{2N}(t_n) \\ \vdots & \vdots & & \vdots \\ p_{N1}(t_n) & p_{N2}(t_n) & \ldots & p_{NN}(t_n) \end{bmatrix}$$

Each matrix $P(t_n)$, $n=0,1,\ldots$, is naturally a stochastic matrix, the elements fulfill conditions

(8) $0 \leq p_{ij}(t_n) \leq 1,\ 1 \leq i,j \leq N,\ n=0,1,\ldots$

and

(9) $\sum_{j=1}^{N} p_{ij}(t_n) = 1,\ i=1,2,\ldots,N,\ n=0,1,\ldots$

Markov processes are usually illustrated by graphs, to each transition probability matrix corresponds a transition diagram (cf. Fig.1.). The nodes of the graph correspond to the states of the phenomenon and the branches between the nodes correspond to the transitions and the transition probabilities.

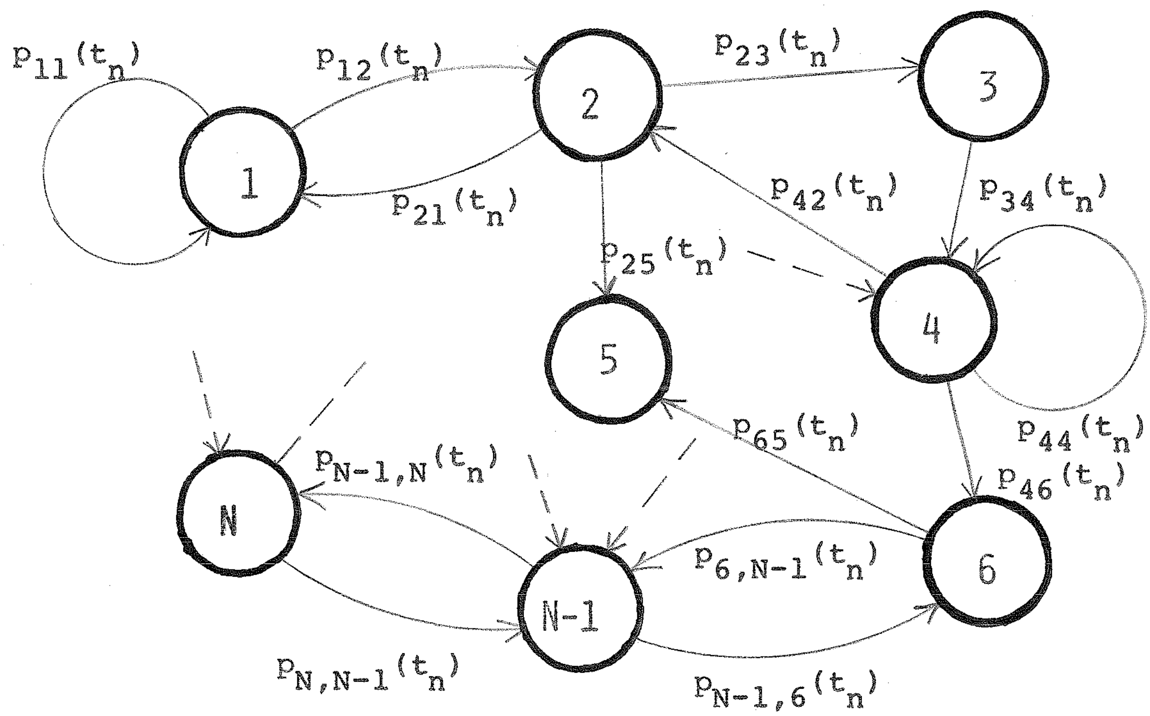

Figure 1. Transition diagram corresponding to a Markov process at time t_n

Later on, when we turn to system theoretical considerations of the Markov process, the state of the system has quite a special meaning. In order to avoid confusion we now give up using the term state in the previous sense and call this concept a node in the following sections. We can state arguments for this term

The multistep transition probabilities describe the transition of the process from one node to another within a given time interval. When we want information merely of which node the process at any given time, independently of the nodes previously occupied by the process, occupies, we use node probabilities

(16) $\pi_i(t_n) = P\{X(t_n)=i\}$, $i=1,2,\ldots,N$, $n=0,1,\ldots$

The vector of these node probabilities is denoted by

(17) $\Pi(t_n) = [\pi_1(t_n),\pi_2(t_n),\ldots,\pi_N(t_n)]$, $n=0,1,\ldots$

The node probabilities at time t_n can be determined, when the node probabilities at time t_0, and the transition probabilities at times $t_0,t_1,\ldots,t_{n-1}$ or, alternatively, the multistep transition probabilities $\phi_{ij}(t_0,t_n)$, $1\leq i,j\leq N$, are known:

(18) $\Pi(t_n) = \Pi(t_0)\Phi(t_0,t_n) = \Pi(t_0)P(t_0)\cdots P(t_{n-1})$, $n=1,2,\ldots$

3. SYSTEMS ANALYSIS OF MARKOV PROCESSES

3.1. Set theoretical basic concepts in system theory

Figure 2 shows a graphical representation of a system, the system diagram:

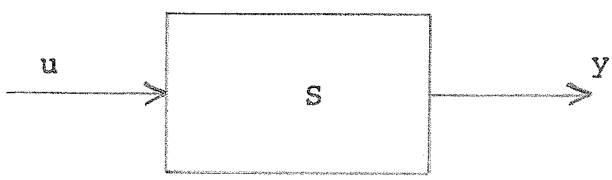

Figure 2. A system

In the diagram S is the system, u is the input and y the output.

The system is uniquely defined, when the set $U = \{u\}$ of all the possible inputs, the set $Y = \{y\}$ of all the possible outputs and a relation S, that gives the inputs and outputs which within this system can be in relation with each others, are known. In set theory terms the system is so a relation

(19) $S = \{(u,y) \mid u \in U, y \in Y, u \to y\}$,

where notation $u \to y$ means that y can appear as an output when u is the input. In Fig.3 y_k is one of the possible outputs y, when input u_i is introduced into the system. All other possible outputs corresponding to u_i are shown with broken lines.

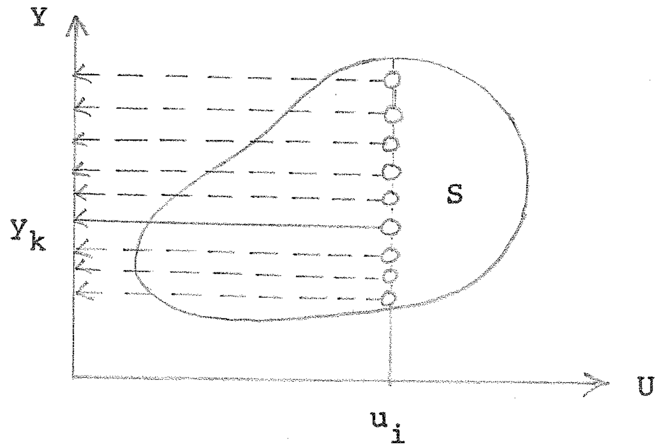

Figure 3. System as a relation

Now we proceed by constructing these basic quantities for the Markov process. Let us assume that the process is observed at discrete time instants $t_0, t_1, \ldots, t_T$, so that the <u>observation period</u> is $\mathcal{T} = [t_0, t_T]$. As the input of the system we choose a sequence of the process' transition probability matrices:

(20) $u = [P(t_0), P(t_1), \ldots, P(t_{T-1})]$,

where each $P(t_i)$, $i=0,1,\ldots,T-1$, is a NxN stochastic matrix in accordance with conditions (8) and (9). Let $\mathcal{P}$ be the set of the NxN stochastic matrices. Then the set of the inputs is

(21) $U(t_0,t_T) = \{u\} \subset \mathcal{P} x \mathcal{P} x \cdots x \mathcal{P} = \mathcal{P}^T.$

As the output we choose a sequence of node probability vectors:

(22) $y = [\Pi(t_1),\Pi(t_2),\ldots,\Pi(t_T)].$

The set of the outputs is so

(23) $Y(t_0,t_T) = \{y\} \subset \Sigma x \Sigma x \cdots x \Sigma = \Sigma^T,$

where Σ stands for the set of vectors which are of the form

(24) $\Sigma = \{(x_1,x_2,\ldots,x_N) \mid 0 \le x_i \le 1, \sum_{i=1}^{N} x_i = 1\}.$

Evidently $\Sigma \subset R^N$. Output y and input u are related according to equation (18). The system corresponding to a Markov process is thus

(25) $S = \{(u,y) \mid u \in U(t_0,t_T), y \in Y(t_0,t_T),$ (18) holds, when $t_n \in \mathcal{T}\}.$

In relation S one input may have several different outputs. Especially, in the case of the Markov process the output corresponding to an input composed of a sequence of transition probability matrices is not uniquely determined until the initial node probability vector $\Pi(t_0)$ is known. An explanation for the fact that the system with the same input has at different times a different output, is called an initial state of the system. When the initial state has been fixed, the input uniquely determines the output. In the case of the Markov process the initial state is clearly the node probability vector $\Pi(t_0)$ at time t_0.

With a fixed initial state $\Pi^o = \Pi(t_0) \in \Sigma$ equation (18) defines a mapping

(26) $\Gamma(\Pi^{o}) : U(t_0,t_T) \rightarrow Y(t_0,t_T)$,

where

(27) $\Gamma(\Pi^{o}) = \{(u,y) \mid u \in U(t_0,t_T), y \in Y(t_0,t_T)$, u and y obey (18)$\}$.

This mapping is a function, i.e. each $u \in U(t_0,t_T)$ has one and only one $y \in Y(t_0,t_T)$ (which is determined through equation (18)). The function $\Gamma(\Pi^{o})$ has properties

(28) $\Gamma(\Pi^{o}) \subset S, \ \forall \ \Pi^{o} \in \Sigma$

and

(29) $\bigcup_{\Pi^{o} \in \Sigma} \Gamma(\Pi^{o}) = S.$

The set Σ which contains all the possible initial states, is called the initial state space of the system. On the basis of the initial states the set S has become divided into subsets $\Gamma(\Pi^{o})$, each one of which is a function and which together form a covering for S. In each element $\Gamma(\Pi^{o})$ of the covering any input determines an unique output. This can be denoted

(30) $y = \Gamma(\Pi^{o})(u)$.

Function $\Gamma(\Pi^{o})$ is scematically presented in Fig.4:

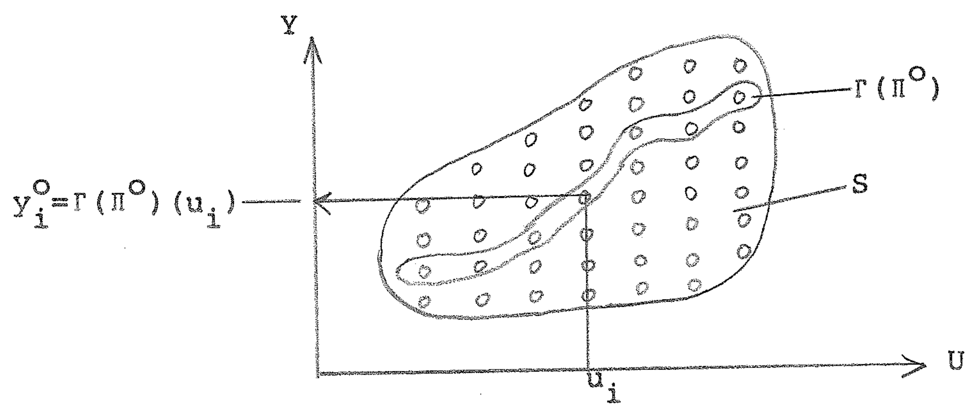

Figure 4. System with a fixed initial state

Now the family of functions

(31) $\mathcal{F} = \{\Gamma(\Pi^{o}) \mid \Pi^{o} \in \Sigma\}$

is on the basis of (28) and (29) a covering of S. For each initial state $\Pi^{o} \in \Sigma$ there is one and only one $\Gamma(\Pi^{o}) \in \mathcal{F}$. Thus Γ can also be interpreted as a function from Σ to $\mathcal{F}$:

(32) $\Gamma : \Sigma \to \mathcal{F}$.

3.2. State of the system at an arbitrary time t_k

In the following our objective is to define and construct a <u>state</u> for the system at an arbitrary time $t_k \in [t_0, t_T]$. We want that this state concept has the same properties as the initial state $\Pi(t_0)$: when the state at time t_k is fixed, so in the interval $[t_k, t_T]$ there is one uniquely determined output-subsequence for each input-subsequence. The function in connection with the state and the covering formed by these functions must also be so constructed that corresponding quantities of the original system have been used.

Denote

(33) $u_{t_m, t_n} = [P(t_m), \ldots, P(t_{n-1})],\ 0 \leq m < n \leq T,$

(34) $y_{t_m, t_n} = [\Pi(t_{m+1}), \ldots, \Pi(t_n)],\ 0 \leq m < n \leq T,$

i.e. u_{t_m, t_n} and y_{t_m, t_n} are the subinput and suboutput in the interval $[t_m, t_n]$ formed out of the input $u \in U(t_0, t_T)$ and the output $y \in Y(t_0, t_T)$, respectively. With the help of S we now form two new relations by taking apart of the input-output pairs

Next we will construct one natural solution for the problem of table 1.

1^{o}. Relation S_{t_k,t_T} has already become defined in equation (36). Clearly the relation has been constructed using the original S, as the objective was.

2^{o}. The state of the system at time t_k is defined as a pair of the initial state Π^{o} and the subinput u_{t_0,t_k} in the interval $[t_0,t_k]$:

(37) $\Pi^{k} = (\Pi^{o},u_{t_0,t_k}),\ \Pi^{o} \in \Sigma,\ u \in U(t_0,t_T).$

Taking definition (33) into account, state Π^{k} becomes

(38) $\Pi^{k} = (\Pi^{o},[P(t_0),P(t_1),\ldots,P(t_{k-1})])$

The state at time t_k is so composed of the node probability vector at time t_0 and of the transition probability matrices at times $t_0,\ldots,t_{k-1}$.

3^{o}. The state space at time t_k is thereafter

(39)
$$\Sigma_k = \{(\Pi^{o},u_{t_0,t_k})\,|\,\Pi^{o}\in\Sigma,\ u\in U(t_0,t_T)\}$$
$$= \{(\Pi^{o},x)\,|\,\Pi^{o}\in\Sigma,\ x\in U(t_0,t_k)\}.$$

4^{o}. Next the function $\Gamma_k(\Pi^{k}) : U(t_k,t_T) \to Y(t_k,t_T)$ will be constructed. Let the state $\Pi^{k} \in \Sigma_k$ be fixed and let r be an arbitrary subinput in the interval $[t_k,t_T]$, i.e.

(40) $r = [P(t_k),\ldots,P(t_{T-1})].$

Γ_k is now defined with the help of Γ as follows:

Knowledge of the node probability vector $\Pi(t_k)$ and the multistep transition probability matrix $\Phi(t_0,t_k)$ is not sufficient for determining the state of the system at time t_k.

In the following some properties of the state will be more closely considered. Let the initial state $\Pi^o \in \Sigma$ and the input $u \in U(t_0,t_T)$ be fixed. For each time point t_k in the observation period $\mathcal{T}$ there is now one uniquely determined state $\Pi^k = (\Pi^o,u_{t_0,t_k}) \in \Sigma_k$. Thus the state can be interpreted as a function

(49) $\gamma : \mathcal{T} \rightarrow \bigcup_{k=1}^{T} \Sigma_k,$

where

(50) $\gamma(t_k) = \Pi^k = (\Pi^o,u_{t_0,t_k}) \in \Sigma_k, \ \forall \ t_k \in \mathcal{T}.$

The state is so a quantity comparable with the output, proceeding in a fully prescribed manner after the initial state of the system is fixed and a given input acts on the system. The state can also be interpreted as a memory of the system. It preserves information of the initial state of the system (Π^o) and information of previous environmental factors of the system (the sub-input u_{t_0,t_k} in the interval $[t_0,t_k]$). Because of this the future behaviour of the system (the output y_{t_k,t_T} in the interval $[t_k,t_T]$) can be derived on the basis of the present state (Π^o,u_{t_0,t_k}) and the future environmental factors of the system (the input u_{t_k,t_T} in the interval $[t_k,t_T]$). So the state at time t_k, Π^k, has wholly equivalent properties with $\mathcal{T}_k = [t_k,t_T]$ as an observation period as the initial state Π^o has with $\mathcal{T} = [t_0,t_T]$ as an observation period. The only thing that must be done is to turn to use the k-indexed quantities given in table 1, when $\mathcal{T}_k$ instead of $\mathcal{T}$ is concerned.

Next we will construct one natural solution for the problem of table 1.

1^{o}. Relation S_{t_k,t_T} has already become defined in equation (36). Clearly the relation has been constructed using the original S, as the objective was.

2^{o}. The state of the system at time t_k is defined as a pair of the initial state Π^{o} and the subinput u_{t_0,t_k} in the interval $[t_0,t_k]$:

(37) $$\Pi^{k} = (\Pi^{o},u_{t_0,t_k}),\ \Pi^{o} \in \Sigma,\ u \in U(t_0,t_T).$$

Taking definition (33) into account, state Π^{k} becomes

(38) $$\Pi^{k} = (\Pi^{o},[P(t_0),P(t_1),\ldots,P(t_{k-1})])$$

The state at time t_k is so composed of the node probability vector at time t_0 and of the transition probability matrices at times $t_0,\ldots,t_{k-1}$.

3^{o}. The state space at time t_k is thereafter

(39) $$\begin{aligned}\Sigma_k &= \{(\Pi^{o},u_{t_0,t_k})\,|\,\Pi^{o}\in\Sigma,\ u\in U(t_0,t_T)\}\\ &= \{(\Pi^{o},x)\,|\,\Pi^{o}\in\Sigma,\ x\in U(t_0,t_k)\}.\end{aligned}$$

4^{o}. Next the function $\Gamma_k(\Pi^{k}) : U(t_k,t_T) \to Y(t_k,t_T)$ will be constructed. Let the state $\Pi^{k} \in \Sigma_k$ be fixed and let r be an arbitrary subinput in the interval $[t_k,t_T]$, i.e.

(40) $$r = [P(t_k),\ldots,P(t_{T-1})].$$

Γ_k is now defined with the help of Γ as follows:

$$\begin{aligned}
\Gamma_k(\Pi^k)(r) &= \Gamma_k(\Pi^o, x)(r)\\
&= \Gamma_k(\Pi^o, [P(t_0), \ldots, P(t_{k-1})])([P(t_k), \ldots, P(t_{T-1})])\\
&= [\Gamma(\Pi^o)([P(t_0), \ldots, P(t_{k-1}), P(t_k), \ldots, P(t_{T-1})])]_{t_k, t_T}\\
&= [\Gamma(\Pi^o)([x, r])]_{t_k, t_T},
\end{aligned} \tag{41}$$

where notation $[\ \ ]_{t_k, t_T}$ means that only the fragment corresponding to the interval $[t_k, t_T]$ is taken out of the output in the brackets, and notation $[x,r]$ stands for a vector, the components of which are the components of x and r put after one another. Because Γ is a function, it follows from definition (41) that Γ_k also is a function. Clearly it also is $\Gamma_k(\Pi^k) \subset [\Gamma(\Pi^o)]_{t_k, t_T}$ and $[\Gamma(\Pi^o)]_{t_k, t_T} \subset S_{t_k, t_T}$. Thus we have

(42) $\Gamma_k(\Pi^k) \subset S_{t_k, t_T}$.

5^o. In the end it will be shown that the family of functions $\mathscr{F}_k = \{\Gamma_k(\Pi^k) \mid \Pi^k \in \Sigma_k\}$ is a covering for S_{t_k, t_T}. In paragraph 4^o we have already shown that all $\Gamma_k(\Pi^k)$'s are functions and subsets of S_{t_k, t_T}. Thus, what we still must show, is, that they cover S_{t_k, t_T}, i.e. that equation

(43) $\bigcup_{\Pi^k \in \Sigma_k} \Gamma_k(\Pi^k) = S_{t_k, t_T}$

holds. Because it is clear on the basis of (42) that relation $\bigcup_{\Pi^k \in \Sigma_k} \Gamma_k(\Pi^k) \subset S_{t_k, t_T}$ holds, it is sufficient for proving (43) to show that also relation $S_{t_k, t_T} \subset \bigcup_{\Pi^k \in \Sigma_k} \Gamma_k(\Pi^k)$ holds. Let (r,p) be an arbitrary element of S_{t_k, t_T}, composed of a subinput and suboutput of S. According to (36) there exist an element $(u,y) \in S$ such that

(44) $(r,p) = (u_{t_k, t_T}, y_{t_k, t_T})$, or

(45) $\quad r = u_{t_k,t_T}$ and $p = y_{t_k,t_T}$.

Because it is $\bigcup_{\Pi^o \in \Sigma} \Gamma(\Pi^o) = S$, there exists $\Pi^o \in \Sigma$ such that $(u,y) \in \Gamma(\Pi^o)$ or $y = \Gamma(\Pi^o)(u)$. Now we choose $x = u_{t_0,t_k}$ and $\Pi^k = (\Pi^o,x)$. Then we have $u = [x,r]$ and

(46) $$\begin{aligned} p = y_{t_k,t_T} &= [\Gamma(\Pi^o)(u)]_{t_k,t_T} = [\Gamma(\Pi^o)([x,r])]_{t_k,t_T} \\ &= \Gamma_k(\Pi^o,x)(r) = \Gamma_k(\Pi^k)(r), \end{aligned}$$

which in other words means $(r,p) \in \Gamma_k(\Pi^k)$. This further indicates that $S_{t_k,t_T} \subset \bigcup_{\Pi^k \in \Sigma_k} \Gamma_k(\Pi^k)$. Combining all the results in paragraph 5^o we can see that family $\mathcal{F}_k = \{\Gamma_k(\Pi^k) \mid \Pi^k \in \Sigma_k\}$ forms a covering for S_{t_k,t_T}.

Family $\mathcal{F}_k$ had Π^k or the pair (Π^o,u_{t_0,t_k}) as a parameter. This pair is called the state of the system S at time t_k (when the initial state Π^o and the input u are fixed). The state of the system at time t_k is so composed of the initial state Π^o and of the "history" of the input in the interval $[t_0,t_k]$. In the case of the Markov process this means the node probability vector $\Pi^o = \Pi(t_0)$ at the initial time t_0 and the sequence $u_{t_0,t_k} = [P(t_0),\ldots,P(t_{k-1})]$ of transition probability matrices. It can further be noted that knowing the state at time t_k means knowing the node probabilities $\pi_i(t_k)$, $i=1,2,\ldots,N$, and multistep transition probabilities $\phi_{ij}(t_0,t_k)$, $i,j=1,2,\ldots,N$, at the same time. For, according to (11) and (18), it is

(47) $$\Pi(t_k) = \Pi(t_0)P(t_0) \cdots P(t_{k-1})$$

and

(48) $$\Phi(t_0,t_k) = P(t_0) \cdots P(t_{k-1}).$$

The inverse result does not, however, hold in the general case.

Knowledge of the node probability vector $\Pi(t_k)$ and the multistep transition probability matrix $\Phi(t_0,t_k)$ is not sufficient for determining the state of the system at time t_k.

In the following some properties of the state will be more closely considered. Let the initial state $\Pi^o \in \Sigma$ and the input $u \in U(t_0,t_T)$ be fixed. For each time point t_k in the observation period $\mathcal{T}$ there is now one uniquely determined state $\Pi^k = (\Pi^o, u_{t_0,t_k}) \in \Sigma_k$. Thus the state can be interpreted as a function

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(50) $\gamma(t_k) = \Pi^k = (\Pi^o, u_{t_0,t_k}) \in \Sigma_k, \ \forall \ t_k \in \mathcal{T}.$

The state is so a quantity comparable with the output, proceeding in a fully prescribed manner after the initial state of the system is fixed and a given input acts on the system. The state can also be interpreted as a memory of the system. It preserves information of the initial state of the system (Π^o) and information of previous environmental factors of the system (the sub-input u_{t_0,t_k} in the interval $[t_0,t_k]$). Because of this the future behaviour of the system (the output y_{t_k,t_T} in the interval $[t_k,t_T]$) can be derived on the basis of the present state (Π^o, u_{t_0,t_k}) and the future environmental factors of the system (the input u_{t_k,t_T} in the interval $[t_k,t_T]$). So the state at time t_k, Π^k, has wholly equivalent properties with $\mathcal{T}_k = [t_k,t_T]$ as an observation period as the initial state Π^o has with $\mathcal{T} = [t_0,t_T]$ as an observation period. The only thing that must be done is to turn to use the k-indexed quantities given in table 1, when $\mathcal{T}_k$ instead of $\mathcal{T}$ is concerned.

transition probability matrices have an exogenous nature. Let the phenomenon under consideration be, for instance, the behaviour of a consumer, when he repeatedly chooses between brands of a given product group. The nodes of the Markov process are in this case different brands, the transitions are choices concerning these brands. The transition probability matrices are now clearly exogenous by nature, for their forms are greatly influenced by several factors originating in the environment of the consumer (price and quality of the brands, fashion, season, consumer information, advertising etc.). These factors represent both controllable and non-controllable quantities.

As the structure of the Markov system it so remains, after the transition probability matrices have been fixed as the input, the law given in the form of equation (18). With fixed initial state and input, this law wholly uniquely determines the output. The structure answering equation (18) comes straightly from the Markovian property of the process. Equation (18) presumes and on the other hand is a consequence of assumption (5): the choice of the next node is only affected by the moment of the choice and by the node occupied by the process at the present time.

3.3. On interpretation of the system presentation

In the foregoing sections a system presentation was formulated for the discrete finite-node Markov process, using principles of general system theory. In this presentation the basic quantities of the system got the following purport (not necessarily the only possible however). For the input of the system we chose a sequence of transition probability matrices. The output got a form of a sequence of node probability vectors. The initial state of the system was defined as the node probability vector at the initial time, the state at an arbitrary time t_k composed of the initial state and of the input advanced into the system before time t_k.

Most of the quantities mentioned above have quite a natural interpretation in accordance with properties of the real phenomenon. The initial state describes the nature of the phenomenon at the initial time t_0. For, the node probability vector $\Pi(t_0)$ determines a distribution for the random variable that gives the running number of the node occupied by the process at time t_0. The output of the system describes changing of this distribution along with time, i.e. it lets one know what the node probabilities at each time point are.

The most difficult quantity to interprete is the input which consists of a sequence of transition probability matrices. Input is usually understood as a purely exogenous quantity, either controllable or non-controllable, while the transition probability matrix has the idea of an endogenous quantity describing the structure of the phenomenon. It is not, however, impossible to find out examples of phenomena, in the description of which the