Cramerin kaava, L6

Nimityksiä

Adjungaatti

Käänteismatriisi

Cramerin kaavat

$$\begin{aligned}
\text{Matriisi} &= A = \begin{pmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{pmatrix} \\
\text{Alimatriisi} &= A_{12} = \begin{pmatrix} 7 & -6 \\ 5 & 0 \end{pmatrix} \\
\text{Minori} &= \det(A_{12}) = \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30 \\
\text{Kofaktori} &= (-1)^{1+2}\det(A_{12}) = -30
\end{aligned}$$

Määritelmä $n \times n$ neliö-matriisin A adjungaatti, $\mathrm{Adj}(A)$, on matriisin A kofaktorikaavion transpoosi.

Helppo tapa tehdä adjungaatti yksinkertaisin askelin, on ensin laskea kaikki minorit. Jos $m_{ij} = \mathrm{Det}(A_{ij})$ on matriisin A paikkaan i,j liittyvä minori, niin

$$\mathrm{Adj}(A) = \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} & \cdots \\ -m_{12} & +m_{22} & -m_{32} & \cdots \\ +m_{13} & -m_{23} & +m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Esimerkki 1: Laske adjungaatti matriisille

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{pmatrix}.$$

Lasketaan ensin minorit

$$m_{11} = \begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} = -12 \quad m_{12} = \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30 \quad m_{13} = \begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} = -34$$

$$m_{21} = \begin{vmatrix} -1 & 3 \\ -2 & 0 \end{vmatrix} = 6 \quad m_{22} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15 \quad m_{23} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1$$

$$m_{31} = \begin{vmatrix} -1 & 3 \\ 4 & -6 \end{vmatrix} = -6 \quad m_{32} = \begin{vmatrix} 2 & 3 \\ 7 & -6 \end{vmatrix} = -33 \quad m_{33} = \begin{vmatrix} 2 & -1 \\ 7 & 4 \end{vmatrix} = 15$$

Adjungaatti 4

$$m_{11} = \begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} = -12 \qquad m_{12} = \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30 \qquad m_{13} = \begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} = -34$$

$$m_{21} = \begin{vmatrix} -1 & 3 \\ -2 & 0 \end{vmatrix} = 6 \qquad m_{22} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15 \qquad m_{23} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1$$

$$m_{31} = \begin{vmatrix} -1 & 3 \\ 4 & -6 \end{vmatrix} = -6 \qquad m_{32} = \begin{vmatrix} 2 & 3 \\ 7 & -6 \end{vmatrix} = -33 \qquad m_{33} = \begin{vmatrix} 2 & -1 \\ 7 & 4 \end{vmatrix} = 15$$

$$\mathrm{Adj}(A) = \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix} = \begin{pmatrix} +(-12) & -(6) & +(-6) \\ -(30) & +(-15) & -(-33) \\ +(-34) & -(1) & +(15) \end{pmatrix}$$

$$= \begin{pmatrix} -12 & -6 & -6 \\ -30 & -15 & 33 \\ -34 & -1 & 15 \end{pmatrix}$$

Lasketaan seuraavaksi matriisin A ja sen adjungaatin $\mathrm{Adj}(A)$ matriisitulo.

$$B = A \cdot \mathrm{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

$$\begin{aligned} b_{11} &= a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathrm{Det}(A) \\ b_{12} &= -a_{11}m_{21} + a_{12}m_{22} - a_{13}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \\ b_{13} &= a_{11}m_{31} - a_{12}m_{32} + a_{13}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = 0 \end{aligned}$$

$$B = A \cdot \mathrm{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

$$\begin{aligned} b_{21} &= a_{21}m_{11} - a_{22}m_{12} + a_{23}m_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \\ b_{22} &= -a_{21}m_{21} + a_{22}m_{22} - a_{23}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathrm{Det}(A) \\ b_{23} &= a_{21}m_{31} - a_{22}m_{32} + a_{23}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = 0 \end{aligned}$$

$$B = A \cdot \mathrm{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

$$b_{31} = a_{31}m_{11} - a_{32}m_{12} + a_{33}m_{13} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{32} = -a_{31}m_{21} + a_{32}m_{22} - a_{33}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{33} = a_{31}m_{31} - a_{32}m_{32} + a_{33}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = \mathrm{Det}(A)$$

Johtopäätös:

$$A \cdot \mathrm{Adj}(A) = \begin{pmatrix} \mathrm{Det}(A) & 0 & 0 \\ 0 & \mathrm{Det}(A) & 0 \\ 0 & 0 & \mathrm{Det}(A) \end{pmatrix} = \mathrm{Det}(A) \cdot I$$

$$\Rightarrow A \left(\frac{\mathrm{Adj}(A)}{\mathrm{Det}(A)} \right) = I$$

Eli

Lause: Jos matriisi A on säännöllinen, eli $\mathrm{Det}(A) \neq 0$, niin

$$A^{-1} = \frac{1}{\mathrm{Det}(A)} \mathrm{Adj}(A)$$

esimerkki 2: Esimerkin 1 matriisille

$$
\begin{aligned}
\mathrm{Det}(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{vmatrix} \\
&= +(2)\begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} + (3)\begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} \\
&= 2\cdot(0-12)+(0+30)+3\cdot(-14-20) = -96
\end{aligned}
$$

Siis

$$
\begin{aligned}
A^{-1} &= \frac{1}{\mathrm{Det}(A)}\cdot \mathrm{Adj}(A) = \frac{1}{-96}\begin{pmatrix} -12 & -6 & -6 \\ -30 & -15 & 33 \\ -34 & -1 & 15 \end{pmatrix} \\
&= \begin{pmatrix} 12/96 & 6/96 & 6/96 \\ 30/96 & 15/96 & -33/96 \\ 34/96 & 1/96 & -15/96 \end{pmatrix}
\end{aligned}
$$

Seuraavaksi ratkaisemme yhtälöryhmän (Oletuksella $D = \mathrm{Det}(A) \neq 0$)

$$
\begin{aligned}
& Ax = b \\
\Leftrightarrow \quad & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
\Leftrightarrow \quad & x = A^{-1} b \\
\Leftrightarrow \quad & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\end{aligned}
$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1 = \frac{1}{D}(m_{11}b_1 - m_{21}b_2 + m_{31}b_3) = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{1}{D}(-m_{12}b_1 + m_{22}b_2 - m_{32}b_3) = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{1}{D}(m_{13}b_1 - m_{23}b_2 + m_{33}b_3) = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Lause (Cramerin kaavat:) Jos kvadraattisen yhtälöryhmän $Ax = b$ kerroinkaavio on säänöllinen ($\mathrm{Det}(A) \neq 0$), niin muuttujan x_k arvo yhtälöryhmän ratkaisussa on

$$x_k = \frac{D_k}{D},$$

missä D on kerroinkaavion determinantti, ja D_k on determinantti kaaviolle, jonka k:s sarake on RHS-vektori b ja muut sarakkeet ovat samoja kuin kerroinkaaviossa A.

Esimerkki: Ratkaistaan yhtälöryhmä

$$\begin{cases} -3x + 2y + z = 0 \\ x - y - z = 0 \\ -x + 3y - z = 6 \end{cases}$$

$$D = \begin{vmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 3 & -1 \end{vmatrix} = -6, \qquad D_1 = \begin{vmatrix} 0 & 2 & 1 \\ 0 & -1 & -1 \\ 6 & 3 & -1 \end{vmatrix} = -6,$$

$$D_2 = \begin{vmatrix} -3 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 6 & -1 \end{vmatrix} = -12, \qquad D_3 = \begin{vmatrix} -3 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 3 & 6 \end{vmatrix} = 6,$$

$$x_1 = \frac{D_1}{D} = \frac{-6}{-6} = 1, \quad x_2 = \frac{D_2}{D} = \frac{-12}{-6} = 2, \quad x_3 = \frac{D_3}{D} = \frac{6}{-6} = -1,$$