Kahden muuttujan Taylorin polynomi

Laskemme ensin osittaisderivaattoja kahden muuttujan potenssisarjalle

$$\begin{aligned} f(x,y) = c_{00} &+ c_{10}(x-a) + c_{01}(y-b) \\ &+ c_{20}(x-a)^2 + c_{11}(x-a)(y-b) + c_{02}(y-b)^2 \\ &+ c_{30}(x-a)^3 + c_{21}(x-a)^2(y-b) + c_{12}(x-a)(y-b)^2 + c_{03}(y-b)^3 \\ &+ \dots \end{aligned}$$

$$f(a,b) = c_{00}$$

$$
\begin{aligned}
f = {} & c_{00} + c_{10}(x-a) + c_{01}(y-b) \\
& + c_{20}(x-a)^2 + c_{11}(x-a)(y-b) + c_{02}(y-b)^2 \\
& + c_{30}(x-a)^3 + c_{21}(x-a)^2(y-b) + c_{12}(x-a)(y-b)^2 + c_{03}(y-b)^3 \\
& + \dots
\end{aligned}
$$

$$
\begin{aligned}
f_x = {} & c_{10} + 2c_{20}(x-a) + c_{11}(y-b) \\
& + 3c_{30}(x-a)^2 + 2c_{21}(x-a)(y-b) + c_{12}(y-b)^2 + \dots
\end{aligned}
$$

$$f_x(a,b) = c_{10}$$

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$$
\begin{aligned}
f &= c_{00} + c_{10}(x-a) + c_{01}(y-b) \\
&\quad + c_{20}(x-a)^2 + c_{11}(x-a)(y-b) + c_{02}(y-b)^2 \\
&\quad + c_{30}(x-a)^3 + c_{21}(x-a)^2(y-b) + c_{12}(x-a)(y-b)^2 + c_{03}(y-b)^3 \\
&\quad + \dots
\end{aligned}
$$

$$
\begin{aligned}
f_y &= c_{01} + c_{11}(x-a) + 2c_{02}(y-b) \\
&\quad + c_{21}(x-a)^2 + c_{12}(x-a)(y-b) + 3c_{03}(y-b)^2 + \dots
\end{aligned}
$$

$$f_y(a,b) = c_{01}$$

$$
\begin{aligned}
f_x &= c_{10} + 2c_{20}(x-a) + c_{11}(y-b) \\
&\quad + 3c_{30}(x-a)^2 + 2c_{21}(x-a)(y-b) + c_{12}(y-b)^2 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
f_{xx} &= 2c_{20} + 2\cdot 3\cdot c_{30}(x-a) + 2c_{21}(y-b) + \ldots \\
f_{xx}(a,b) &= 2c_{20}
\end{aligned}
$$

$$
\begin{aligned}
f_{xy} &= c_{11} + 2c_{21}(x-a) + 2c_{12}(y-b) + \ldots \\
f_{xy}(a,b) &= c_{11}
\end{aligned}
$$

$$
\begin{aligned}
f_y &= c_{01} + c_{11}(x-a) + 2c_{02}(y-b) \\
&\quad + c_{21}(x-a)^2 + c_{12}(x-a)(y-b) + 3c_{03}(y-b)^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
f_{yx} &= c_{11} + 2c_{21}(x-a) + c_{12}(y-b) + \dots \\
f_{yx}(a,b) &= c_{11}
\end{aligned}
$$

$$
\begin{aligned}
f_{yy} &= 2c_{02} + c_{12}(x-a) + 2 \cdot 3 \cdot c_{03}(y-b) + \dots \\
f_{yy}(a,b) &= 2c_{02}
\end{aligned}
$$

Toisen kertaluvun Taylorin polynomi on nyt

$$
\begin{aligned}
P_2(x,y) &= f + f_x \cdot (x-a) + f_y \cdot (y-b) \\
&\quad + \frac{f_{xx}}{2} \cdot (x-a)^2 + f_{xy} \cdot (x-a)(y-b) + \frac{f_{yy}}{2} \cdot (y-b)^2 \\
&= f + \nabla f^T \vec{w} + \frac{1}{2} \vec{w}^T \mathbf{H} \vec{w}
\end{aligned}
$$

missä

$$
\vec{w} = \begin{pmatrix} x-a \\ y-b \end{pmatrix}, \quad \nabla f = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.
$$

P_2:n ääriarvokohta on

$$
\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \mathbf{H}^{-1} \nabla f \qquad \text{(perustelu seraavilla sivuilla)}
$$

P_2: ääriarvokohta 7

Lokaalin ääriarvon välttämätön ehto funktion $(\nabla f = \vec{g})$

$$
\begin{aligned}
z(\vec{w}) &= \nabla f^T \vec{w} + \frac{1}{2}\vec{w}^T \mathbf{H} \vec{w} \\
&= \begin{pmatrix} g_1 & g_2 & g_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\
&= g_1 w_1 + g_2 w_2 + g_3 w_3 + \frac{1}{2} h_{11} w_1^2 + \frac{1}{2} h_{22} w_2^2 + \frac{1}{2} h_{33} w_3^2 + \\
&\quad + h_{12} w_1 w_2 + h_{13} w_1 w_3 + h_{23} w_2 w_3
\end{aligned}
$$

on

$$
\left\{ \begin{array}{ccc} z_{w_1} & = & 0 \\ z_{w_2} & = & 0 \\ z_{w_3} & = & 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{ccc} g_1 + h_{11} w_1 + h_{12} w_2 + h_{13} w_3 & = & 0 \\ g_2 + h_{12} w_1 + h_{22} w_2 + h_{23} w_3 & = & 0 \\ g_3 + h_{13} w_1 + h_{23} w_2 + h_{33} w_3 & = & 0 \end{array} \right.
$$

Eli

$$\begin{cases} g_1 + h_{11}w_1 + h_{12}w_2 + h_{13}w_3 & = & 0 \\ g_2 + h_{12}w_1 + h_{22}w_2 + h_{23}w_3 & = & 0 \\ g_3 + h_{13}w_1 + h_{23}w_2 + h_{33}w_3 & = & 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} h_{11}w_1 + h_{12}w_2 + h_{13}w_3 & = & -g_1 \\ h_{12}w_1 + h_{22}w_2 + h_{23}w_3 & = & -g_2 \\ h_{13}w_1 + h_{23}w_2 + h_{33}w_3 & = & -g_3 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = -\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

$$\Leftrightarrow \mathbf{H}\vec{w} = -\nabla f$$

$$\Leftrightarrow \vec{w} = -\mathbf{H}^{-1}\nabla f$$

Etsitään funktion
$f(x,y) = -x^2 + xy - 2y^2 + 5x - 15y + 0.01x^3$ ääriarvoa.
Lasketaan pisteeseen $(15,-15)$ kehitetyn toisen kertaluvun Taylorin polynomin ääriarvokohta.
Tehdään siitä uusi kehityskeskus ja toistetaan tätä kunnes gradientti on lyhyt tai askeleita on otettu sata.

Esimerkki

Koodi:

```
/* f(x,y) = -x^2 + x*y - 2*y^2 + 5*x - 15*y  + 0.01*x^3
w = zeros(2,100);
w(:,1) = [15,-15]'; k=1;
do
  x = w(1,k);
  y = w(2,k);
  fx = -2*x + y + 5 + 0.03*x^2;
  fy = x -4*y -10;
  fxx = -2 + 0.06*x;
  fxy = 1;
  fyx = 1;
  fyy = -4;
  g = [fx fy]';
  H = [fxx fxy; fyx fyy];
  w(:,k+1) = w(:,k) - inv(H)*g;
  k = k+1;
until (norm(g)<0.001) | (k>100)
k=k-1;
w(:,1:k)'
```

Ohjelman output:

```
>> qpesim2
ans =
   15.00000  -15.00000
   -5.00000   -3.75000
    0.85366   -2.28659
    1.45877   -2.13531
    1.46538   -2.13365
```

Yleisemmin

$$f(x) = f(a,b) + \sum_{k+p>0} \frac{\left(\frac{\partial^k}{\partial x^k}\frac{\partial^p}{\partial y^p}\right) f}{k! \cdot p!} \cdot (x-a)^k (y-b)^p$$