Vaasan yliopisto, kevät 2015 / ORMS1010

Matemaattinen Analyysi

1. harjoitus, viikko 10

R1	ma	16–18	D115
R2	ke	12–14	B209

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1. Laske $\langle \boldsymbol{u}_1 \mid \boldsymbol{u}_2 \rangle$, $\langle \boldsymbol{u}_1 \mid \boldsymbol{u}_3 \rangle$, $\langle \boldsymbol{u}_2 \mid \boldsymbol{u}_3 \rangle$, $\|\boldsymbol{u}_1\|$, $\|\boldsymbol{u}_2\|$ ja $\|\boldsymbol{u}_3\|$, kun

$$\boldsymbol{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\vec{u}_1 \cdot \vec{u}_2 = 2\cdot 0 + (-1)\cdot 1 + 1\cdot(-2) + 3\cdot 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 2\cdot 1 + (-1)\cdot 2 + 1\cdot 1 + 3\cdot 0 = 1$$

$$\vec{u}_2 \cdot \vec{u}_3 = 0\cdot 1 + 1\cdot 2 + (-2)\cdot 1 + 1\cdot 0 = 0$$

$$\|\bar{u}_1\| = \sqrt{2^2 + (-1)^2 + 1^2 + 3^2} = \sqrt{4+1+1+9} = \sqrt{15}$$

$$\|\bar{u}_2\| = \sqrt{0^2 + 1^2 + (-2)^2 + 1^2} = \sqrt{0+1+4+1} = \sqrt{6}$$

$$\|\bar{u}_3\| = \sqrt{1^2 + 2^2 + 1^2 + 0^2} = \sqrt{1+4+1+0} = \sqrt{6}$$

2. Ratkaise x, y ja z yhtälöryhmästä

$$\begin{cases} x + y + z = 1 \\ \quad\;\; y + z = 0 \\ x - y + z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \cdot(-1) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right) \Leftrightarrow \begin{cases} x + y + z = 1 & (1) \\ y + z = 0 & (2) \\ -2y = -1 & (3) \end{cases}$$

$$(3) \rightarrow y = \frac{1}{2}$$

$$(2) \rightarrow \frac{1}{2} + z = 0 \Rightarrow z = -\frac{1}{2}$$

$$(1) \rightarrow x + \frac{1}{2} - \frac{1}{2} = 1 \rightarrow x = 1$$

Vi $x = 1$, $y = \frac{1}{2}$, $z = -\frac{1}{2}$

3. Pystyykö vektorin $\boldsymbol{x} = (2 \quad -1 \quad 0 \quad 5)^T$ lausumaan tehtävän 1 vektorien $\mathbf{u}_1$, $\boldsymbol{u}_2$ ja $\boldsymbol{u}_3$ lineaarikombinaationa? Siis onko olemassa reaaliluvut $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ siten, että

$$\boldsymbol{x} = \lambda_1 \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2 + \lambda_3 \boldsymbol{u}_3$$

$$\lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_3 = \bar{x}$$

$$\Leftrightarrow \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 4 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2\lambda_1 + \lambda_3 = 2 \\ -\lambda_1 + \lambda_2 + 2\lambda_3 = -1 \\ \lambda_1 - 2\lambda_2 + \lambda_3 = 0 \\ 3\lambda_1 + 4\lambda_2 = 5 \end{cases} \text{ vaihda}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 1 & 2 & -1 \\ 2 & 0 & 1 & 2 \\ 3 & 4 & 0 & 5 \end{array}\right) \begin{array}{l} \cdot 1 \quad \cdot 2 \quad \cdot 3 \\ + \\ - \\ - \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & -4 & -1 & 2 \\ 0 & 10 & -3 & 5 \end{array}\right) \begin{array}{l} \cdot 4 \quad \cdot 10 \\ - \\ + \end{array} \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -13 & 6 \\ 0 & 0 & 27 & -5 \end{array}\right) \begin{array}{l} \cdot 2 \\ + \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -13 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right) \begin{array}{l} + \\ \cdot 13 \end{array} \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 97 \\ 0 & 0 & 1 & 7 \end{array}\right) \leftarrow \text{VV}$$

V: ei ole olemassa reaalilukuja $\lambda_1, \lambda_2, \lambda_3$ siten, että

$$\bar{x} = \lambda_1 \bar{u}_1 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_3$$

4. Lausu vektori $s \in \mathbb{R}^3$ kannassa $W = \{w_1, w_2, w_3\}$, kun

$$s = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, w_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_1 \bar{w}_1 + \lambda_2 \bar{w}_2 + \lambda_3 \bar{w}_3 = \bar{s}$$

$$\Leftrightarrow \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 \quad\quad + 2\lambda_3 = 2 \\ \lambda_1 - \lambda_2 + \lambda_3 = 0 \\ \lambda_1 + \lambda_2 \quad\quad = -1 \end{cases} \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{array}\right) \cdot 1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 1 & -2 & -3 \end{array}\right) \cdot 1 \cdot (-1) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -5 \end{array}\right)$$

$$\Leftrightarrow \begin{cases} \lambda_1 \quad\quad 2\lambda_3 = 2 & (1) \\ \lambda_2 + \lambda_3 = 2 & (2) \\ -3\lambda_3 = -5 & (3) \end{cases}$$

$$(3) \rightarrow \lambda_3 = \frac{5}{3}$$

$$(2) \rightarrow \lambda_2 + \frac{5}{3} = 2 \Rightarrow \lambda_2 = \frac{1}{3}$$

$$\lambda_3 = \frac{1}{3} \;\&\; (1) \Rightarrow \lambda_1 + 2 \cdot \frac{5}{3} = 2 \Rightarrow \lambda_1 = \frac{-4}{3}$$

$$\therefore \quad \vec{s} = \frac{-4}{3} \bar{w}_1 + \frac{1}{3} \bar{w}_2 + \frac{5}{3} \bar{w}_3$$

Tarkistus:

$$-\frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -4/3 \\ -4/3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 10/3 \\ 5/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

5. Onko vektorikolmikko

$$U = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$$

lineaarisesti riippumaton? Onko se $\mathbb{R}^3$:n kanta? (Käytä determinanttia.)

Vapaus?

$$\lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 \\ 1 & -2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right) \begin{array}{c} \cdot 1 \cdot 2 \\ + \\ + \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 2 & 9 & 0 \end{array}\right) \begin{array}{c} \\ \cdot 2 \\ + \end{array} \sim \left(\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 21 & 0 \end{array}\right) \Leftrightarrow \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{array}$$

$\Rightarrow \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ on vapaa

Virittääkö $\mathbb{R}^3$:n olkoon $\bar{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$

$$\lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\to \left(\begin{array}{ccc|c} -1 & 1 & 4 & w_2 \\ 1 & -2 & 2 & w_3 \\ 2 & 0 & 1 & w_1 \end{array}\right) \begin{array}{c} \cdot 1 \cdot 2 \\ + \\ + \end{array} \sim \left(\begin{array}{ccc|c} -1 & 1 & 4 & w_2 \\ 0 & -1 & 6 & w_2 + w_3 \\ 0 & 2 & 9 & w_1 + 2w_2 \end{array}\right) \begin{array}{c} \\ \cdot 2 \\ + \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 4 & w_2 \\ 0 & -1 & 6 & w_2 + w_3 \\ 0 & 0 & 21 & w_1 + 4w_2 + 2w_3 \end{array}\right)$$

$\therefore$ λ:t saadaan aina ratkaistua

$\rightarrow$ $\bar{u}_1, \bar{u}_2, \bar{u}_3$ virittää $\mathbb{R}^3$:n

$\therefore$ $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ on $\mathbb{R}^3$:n kanta