Vaasan yliopisto, kevät 2009

Talousmatematiikan perusteet, ORMS1030

10. harjoitus, viikko 14 (30.3.–3.4.09)

R1	ma	10–12	D115	R4	to	08–10	D115
R2	ma	14–16	D115	R5	to	14–16	**D115**
R3	ti	08–10	D115	R6	pe	08–10	D102
				R7	pe	12–14	**D115**

Olkoon tutkittavina matriisit

$$\boldsymbol{M} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{ja} \quad \boldsymbol{N} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}.$$

1. Laske a) determinantti b) transpoosi ja c) käänteismatriisi matriisille $\boldsymbol{M}$.

a) $$\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 1 \cdot (3 \cdot 1 - (-1) \cdot 0) - 2 \cdot (2 \cdot 1 - (-1) \cdot 0) + 0 \cdot (2 \cdot 0 - 3 \cdot 0)$$

$$= 1 \cdot 3 - 2 \cdot 2 + 0$$

$$= -1$$

b) $$M^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

c I) $$M_{11} = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3 \quad M_{12} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2 \quad M_{13} = \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{21} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = -2 \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1 \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$M^{-1} = \frac{1}{\det(M)} \operatorname{Adj}(M) = \frac{1}{\det(M)} \begin{pmatrix} +M_{11} & -M_{21} & +M_{31} \\ -M_{12} & +M_{22} & -M_{32} \\ +M_{13} & -M_{23} & +M_{33} \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} +3 & -2 & +(-2) \\ -2 & +1 & -(-1) \\ +0 & -0 & +(-1) \end{pmatrix} = \begin{pmatrix} -3 & 2 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

c II)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) -2 \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ +2 \ \cdot(-1) \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -1 \quad +2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 2 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow M^{-1} = \begin{pmatrix} -3 & 2 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Määritä rivioperaatioiden avulla käänteismatriisi matriisille $\boldsymbol{N}$.

$$(N \mid I) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) -3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \begin{array}{l} \\ +\frac{1}{2} \ :(-2) \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ +\frac{1}{2} \quad -\frac{1}{2} \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \Rightarrow N^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

$$\sim (I \mid N^{-1})$$

3. Määritä adjungaatin avulla käänteismatriisi matriisille $\boldsymbol{N}$.

$$\det(N) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} = +0 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 0 - 0 + 1 \cdot (1 \cdot 1 - 1 \cdot 3)$$

$$= -2$$

$$N_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \qquad N_{12} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1 \qquad N_{13} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -2$$

$$N_{21} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad N_{22} = \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3 \qquad N_{23} = \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} = 0$$

$$N_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad N_{32} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad N_{33} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$N^{-1} = \frac{1}{\det(N)} \begin{pmatrix} +N_{11} & -N_{21} & +N_{31} \\ -N_{12} & +N_{22} & -N_{32} \\ +N_{13} & -N_{23} & +N_{33} \end{pmatrix}$$

$$= \frac{1}{-2} \begin{pmatrix} +1 & -(-1) & +(-1) \\ -1 & +(-3) & -(-1) \\ +(-2) & -0 & +0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & -1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

4. Laske determinantit

a) $\begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix}$ b) $\begin{vmatrix} 2 & 0 & 1 \\ 5 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix}$ c) $\begin{vmatrix} 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

a) $\begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} = 2\cdot 1 - (-1)\cdot(-3) = -1$

b) $\begin{vmatrix} 2 & 0 & 1 \\ 5 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 5 & 0 \\ 3 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix}$

$= 2(1-0) - 0 + 1\cdot(5-3)$

$= 2 - 0 + 2$

$= 4$

c) $\begin{vmatrix} 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \begin{matrix} \\ -2 \\ \\ \\ \\ \end{matrix} = \begin{vmatrix} 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & -3 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

$= 2\cdot 1\cdot 3\cdot 1\cdot 2 = 12$

5. Ratkaise Cramerin kaavoilla yhtälöryhmä

$$\begin{cases} x - 2y + z = -7 \\ y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 3 & 9 & -20 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 0 & -2 \\ 3 & -20 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 9 \end{vmatrix}$$

$$= 1 \cdot (-20 + 18) + 2(0 + 6) + 1 \cdot (0 - 3)$$

$$= 7$$

$$D_1 = \begin{vmatrix} -7 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 9 & -20 \end{vmatrix} = +(-7) \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} + 0 - 0 = 14$$

$$D_2 = \begin{vmatrix} 1 & -7 & 1 \\ 0 & 0 & -2 \\ 3 & 0 & -20 \end{vmatrix} = -(-7) \begin{vmatrix} 0 & -2 \\ 3 & -20 \end{vmatrix} + 0 - 0 = 42$$

$$D_3 = \begin{vmatrix} 1 & -2 & -7 \\ 0 & 1 & 0 \\ 3 & 9 & 0 \end{vmatrix} = +(-7) \begin{vmatrix} 0 & 1 \\ 3 & 9 \end{vmatrix} = 21$$

$$x = \frac{D_1}{D} = \frac{14}{7} = 2$$

$$y = \frac{D_2}{D} = \frac{42}{7} = 6$$

$$z = \frac{D_3}{D} = \frac{21}{7} = 3$$

Vastaus: $x = 2, \; y = 6, \; z = 3$

Tarkistus

$$2 - 2 \cdot 6 + 3 = -7 \quad ✓$$

$$6 - 2 \cdot 3 = 0 \quad ✓$$

$$3 \cdot 2 + 9 \cdot 6 - 20 \cdot 3 = 0 \quad ✓$$

6. Etsi jokin ei-triviaali ratkaisu yhtälöryhmälle

$$\begin{cases} 3x - 3y + 4z = 0 \\ \quad\;\; y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 9 & -20 & 0 \end{array}\right) \begin{array}{c} -1 \\ \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 12 & -24 & 0 \end{array}\right) \begin{array}{c} \\ -12 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} 3x - 3y = -4z & (1) \\ \qquad\;\; y = 2z & (2) \end{cases}$$

Valitaan $z = 1$, silloin

$(2) \rightarrow y = 2 \cdot 1 = 2$

$(1) \rightarrow 3x - 3 \cdot 2 = -4 \cdot 1 \rightarrow x = 2/3$

Vastaus $\underline{x = 2/3, \; y = 2, \; z = 1}$

(Jos valitaan $z = 3$, niin

$(2) \rightarrow y = 2 \cdot 3 = 6$

$(1) \rightarrow 3x - 3 \cdot 6 = -4 \cdot 3 \rightarrow x = 2$

Siis $x = 2, \; y = 6, \; z = 3$)

7. Ratkaise muuttujan x arvo Cramerin kaavoilla yhtälöryhmästä

$$\begin{cases} x - 2y + z = 0 \\ \phantom{x - {}} 4y - 2z = 5 \\ ax - y + z = 0 \end{cases}, \qquad a \in \mathbb{R}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 4 & -2 \\ a & -1 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 4 & -2 \\ -1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} + a \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 1(4-2) - 0 + a(4-4)$$

$$= 2$$

$$D_1 = \begin{vmatrix} 0 & -2 & 1 \\ 5 & 4 & -2 \\ 0 & -1 & 1 \end{vmatrix} = +0 - 5 \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} + 0$$

$$= -5(-2-(-1)) = +5$$

$$x = \frac{D_1}{D} = \frac{+5}{2} = +2{,}5$$