

# Neliömatriisin adjungaatti, L24

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$$\text{Matriisi} = A = \begin{pmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{pmatrix}$$

$$\text{Alimatriisi} = A_{12} = \begin{pmatrix} 7 & -6 \\ 5 & 0 \end{pmatrix}$$

$$\text{Minori} = \det(A_{12}) = \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30$$

$$\text{Kofaktori} = (-1)^{1+2} \det(A_{12}) = -30$$

**Määritelmä**  $n \times n$  neliö-matriisin  $A$  **adjungaatti**,  $\text{Adj}(A)$ , on matriisin  $A$  kofaktorikaavion transpoosi.

Helppo tapa tehdä adjungaatti yksinkertaisin askelin, on ensin laskea kaikki minorit. Jos  $m_{ij} = \text{Det}(A_{ij})$  on matriisin  $A$  paikkaan  $i, j$  liittyvä minori, niin

$$\text{Adj}(A) = \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} & \cdots \\ -m_{12} & +m_{22} & -m_{32} & \cdots \\ +m_{13} & -m_{23} & +m_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Esimerkki 1:** Laske adjungaatti matriisille

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{pmatrix}.$$

Lasketaan ensin minorit

$$\begin{aligned} m_{11} &= \begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} = -12 & m_{12} &= \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30 & m_{13} &= \begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} = -34 \\ m_{21} &= \begin{vmatrix} -1 & 3 \\ -2 & 0 \end{vmatrix} = 6 & m_{22} &= \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15 & m_{23} &= \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1 \\ m_{31} &= \begin{vmatrix} -1 & 3 \\ 4 & -6 \end{vmatrix} = -6 & m_{32} &= \begin{vmatrix} 2 & 3 \\ 7 & -6 \end{vmatrix} = -33 & m_{33} &= \begin{vmatrix} 2 & -1 \\ 7 & 4 \end{vmatrix} = 15 \end{aligned}$$

$$m_{11} = \begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} = -12$$

$$m_{12} = \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} = 30$$

$$m_{13} = \begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} = -34$$

$$m_{21} = \begin{vmatrix} -1 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$m_{22} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15$$

$$m_{23} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1$$

$$m_{31} = \begin{vmatrix} -1 & 3 \\ 4 & -6 \end{vmatrix} = -6$$

$$m_{32} = \begin{vmatrix} 2 & 3 \\ 7 & -6 \end{vmatrix} = -33$$

$$m_{33} = \begin{vmatrix} 2 & -1 \\ 7 & 4 \end{vmatrix} = 15$$

$$\begin{aligned} \text{Adj}(A) &= \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix} = \begin{pmatrix} +(-12) & -(6) & +(-6) \\ -(30) & +(-15) & -(-33) \\ +(-34) & -(1) & +(15) \end{pmatrix} \\ &= \begin{pmatrix} -12 & -6 & -6 \\ -30 & -15 & 33 \\ -34 & -1 & 15 \end{pmatrix} \end{aligned}$$

Lasketaan seuraavaksi matriisin  $A$  ja sen adjungaatin  $\text{Adj}(A)$  matriisitulo.

$$B = A \cdot \text{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

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$$b_{11} = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \text{Det}(A)$$

$$b_{12} = -a_{11}m_{21} + a_{12}m_{22} - a_{13}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{13} = a_{11}m_{31} - a_{12}m_{32} + a_{13}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$B = A \cdot \text{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

$$b_{21} = a_{21}m_{11} - a_{22}m_{12} + a_{23}m_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{22} = -a_{21}m_{21} + a_{22}m_{22} - a_{23}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \text{Det}(A)$$

$$b_{23} = a_{21}m_{31} - a_{22}m_{32} + a_{23}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = 0$$

$$B = A \cdot \text{Adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} +m_{11} & -m_{21} & +m_{31} \\ -m_{12} & +m_{22} & -m_{32} \\ +m_{13} & -m_{23} & +m_{33} \end{pmatrix}$$

$$b_{31} = a_{31}m_{11} - a_{32}m_{12} + a_{33}m_{13} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{32} = -a_{31}m_{21} + a_{32}m_{22} - a_{33}m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$b_{33} = a_{31}m_{31} - a_{32}m_{32} + a_{33}m_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = \text{Det}(A)$$



Johtopäätös:

$$A \cdot \text{Adj}(A) = \begin{pmatrix} \text{Det}(A) & 0 & 0 \\ 0 & \text{Det}(A) & 0 \\ 0 & 0 & \text{Det}(A) \end{pmatrix} = \text{Det}(A) \cdot I$$

$$\Rightarrow A \left( \frac{\text{Adj}(A)}{\text{Det}(A)} \right) = I$$

Eli

**Lause:** Jos matriisi  $A$  on säännöllinen, eli  $\text{Det}(A) \neq 0$ , niin

$$A^{-1} = \frac{1}{\text{Det}(A)} \text{Adj}(A)$$

**esimerkki 2:** Esimerkin 1 matriisille

$$\begin{aligned}\text{Det}(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 7 & 4 & -6 \\ 5 & -2 & 0 \end{vmatrix} \\ &= + (2) \begin{vmatrix} 4 & -6 \\ -2 & 0 \end{vmatrix} - (-1) \begin{vmatrix} 7 & -6 \\ 5 & 0 \end{vmatrix} + (3) \begin{vmatrix} 7 & 4 \\ 5 & -2 \end{vmatrix} \\ &= 2 \cdot (0 - 12) + (0 + 30) + 3 \cdot (-14 - 20) = -96\end{aligned}$$

Siis

$$\begin{aligned}A^{-1} &= \frac{1}{\text{Det}(A)} \cdot \text{Adj}(A) = \frac{1}{-96} \begin{pmatrix} -12 & -6 & -6 \\ -30 & -15 & 33 \\ -34 & -1 & 15 \end{pmatrix} \\ &= \begin{pmatrix} 12/96 & 6/96 & 6/96 \\ 30/96 & 15/96 & -33/96 \\ 34/96 & 1/96 & -15/96 \end{pmatrix}\end{aligned}$$

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Seuraavaksi ratkaisemme yhtälöryhmän (Oletuksella

$$D = \text{Det}(A) \neq 0)$$

$$Ax = b$$

$$\Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Leftrightarrow x = A^{-1}b$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1 = \frac{1}{D}(m_{11}b_1 - m_{21}b_2 + m_{31}b_3) = \frac{1}{D} \begin{vmatrix} \textcolor{red}{b}_1 & a_{12} & a_{13} \\ \textcolor{red}{b}_2 & a_{22} & a_{23} \\ \textcolor{red}{b}_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{1}{D}(-m_{12}b_1 + m_{22}b_2 - m_{32}b_3) = \frac{1}{D} \begin{vmatrix} a_{11} & \textcolor{red}{b}_1 & a_{13} \\ a_{21} & \textcolor{red}{b}_2 & a_{23} \\ a_{31} & \textcolor{red}{b}_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{1}{D}(m_{13}b_1 - m_{23}b_2 + m_{33}b_3) = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & \textcolor{red}{b}_1 \\ a_{21} & a_{22} & \textcolor{red}{b}_2 \\ a_{31} & a_{32} & \textcolor{red}{b}_3 \end{vmatrix}$$

**Lause (Cramerin kaavat:)** Jos kvadraattisen yhtälöryhmän  $Ax = b$  kerroinkaavio on säännöllinen ( $\text{Det}(A) \neq 0$ ), niin muuttujan  $x_k$  arvo yhtälöryhmän ratkaisussa on

$$x_k = \frac{D_k}{D},$$

missä  $D$  on kerroinkaavion determinantti, ja  $D_k$  on determinantti kaaviolle, jonka  $k$ :s sarake on RHS-vektori  $b$  ja muut sarakkeet ovat samoja kuin kerroinkaaviossa  $A$ .

**Esimerkki:** Ratkaistaan yhtälöryhmä

$$\begin{cases} -3x + 2y + z = 0 \\ x - y - z = 0 \\ -x + 3y - z = 6 \end{cases}$$

$$D = \begin{vmatrix} -3 & 2 & 1 \\ 1 & -1 & -1 \\ -1 & 3 & -1 \end{vmatrix} = -6, \quad D_1 = \begin{vmatrix} 0 & 2 & 1 \\ 0 & -1 & -1 \\ 6 & 3 & -1 \end{vmatrix} = -6,$$

$$D_2 = \begin{vmatrix} -3 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 6 & -1 \end{vmatrix} = -12, \quad D_3 = \begin{vmatrix} -3 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 3 & 6 \end{vmatrix} = 6,$$

$$x_1 = \frac{D_1}{D} = \frac{-6}{-6} = 1, \quad x_2 = \frac{D_2}{D} = \frac{-12}{-6} = 2, \quad x_3 = \frac{D_3}{D} = \frac{6}{-6} = -1,$$