

1. Määritä rivioperaatioiden avulla käänteismatriisi matriisille

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} (A|I) &\sim \begin{pmatrix} 3 & 1 & 2 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 2 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 3 & 1 & 2 & | & 1 & 0 & 0 \\ 2 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 1 & -3 & 0 \\ 0 & -2 & 1 & | & 0 & -2 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} R_3 + 2R_2 \\ R_2 \cdot \frac{1}{2} \end{matrix}} \begin{pmatrix} 1 & 0 & 1 & | & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & | & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{pmatrix} \xrightarrow{\begin{matrix} R_1 - R_3 \\ R_2 - R_3 \end{matrix}} \begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & | & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 1 & | & 1 & -1 & -1 \end{pmatrix} \rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -1 & -1 \end{pmatrix} \end{aligned}$$

2. Määritä adjungaatin avulla käänteismatriisi matriisille

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\det(A) = 3 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 3 - 1 + 2(0 - 2) = -2$$

$$|A_{11}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad |A_{12}| = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \quad |A_{13}| = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$|A_{21}| = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \quad |A_{22}| = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1 \quad |A_{23}| = \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$|A_{31}| = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \quad |A_{32}| = \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -2 \quad |A_{33}| = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} +|A_{11}| & -|A_{21}| & +|A_{31}| \\ -|A_{12}| & +|A_{22}| & -|A_{32}| \\ +|A_{13}| & -|A_{23}| & +|A_{33}| \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} +(1) & -(1) & +(-2) \\ -(1) & +(-1) & -(-2) \\ +(-2) & -(-2) & +(2) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} -x + 7y + 5z = 3 & (1) \\ y + z = -1 & (2) \\ z = -2 & (3) \end{cases}$$

$$(3) \rightarrow z = -2$$

$$(2) \rightarrow y + (-2) = -1 \rightarrow y = 1$$

$$(1) \rightarrow -x + 7 \cdot 1 + 5(-2) = 3$$

$$-x = 3 - 7 + 10$$

$$x = -6$$

$$V: \begin{cases} x = -6 \\ y = 1 \\ z = -2 \end{cases} \therefore \vec{x} = \begin{pmatrix} -6 \\ 1 \\ -2 \end{pmatrix}$$

4. Ratkaise Cramerin kaavoilla yhtälöryhmä

$$\begin{cases} x - 2y + z = -7 \\ y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 3 & 9 & -20 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} - 0 + 3 \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}$$

$$= (-20 - (-18)) - 0 + 3(4 - 1)$$

$$= -2 + 9 = 7$$

$$D_1 = \begin{vmatrix} -7 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 9 & -20 \end{vmatrix} = +(-7) \begin{vmatrix} 1 & -2 \\ 9 & -20 \end{vmatrix} = -7(-20 - (-18))$$

$$= 14$$

$$D_2 = \begin{vmatrix} 1 & -7 & 1 \\ 0 & 0 & -2 \\ 3 & 0 & -20 \end{vmatrix} = -(-7) \begin{vmatrix} 0 & -2 \\ 3 & -20 \end{vmatrix} = 7(0 - (-6)) = 42$$

$$D_3 = \begin{vmatrix} 1 & -2 & -7 \\ 0 & 1 & 0 \\ 3 & 9 & 0 \end{vmatrix} = +1 \cdot \begin{vmatrix} 1 & -7 \\ 3 & 0 \end{vmatrix} = 21$$

$$\begin{cases} x = \frac{D_1}{D} = 2 \\ y = \frac{D_2}{D} = 6 \\ z = \frac{D_3}{D} = 3 \end{cases}$$

5. Etsi jokin ei-triviaali ratkaisu yhtälöryhmälle

$$\begin{cases} 3x - 3y + 4z = 0 \\ y - 2z = 0 \\ 3x + 9y - 20z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} \textcircled{3} & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 9 & -20 & 0 \end{array} \right) \begin{array}{l} \cdot 1 \\ \\ \leftarrow - \end{array} \sim \left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & \textcircled{1} & -2 & 0 \\ 0 & 12 & -24 & 0 \end{array} \right) \begin{array}{l} \\ \cdot 12 \\ \leftarrow - \end{array}$$

$$\left(\begin{array}{ccc|c} 3 & -3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \begin{cases} 3x - 3y + 4z = 0 \\ y - 2z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 3x - 3y = -4z & (1) \\ y = 2z & (2) \end{cases}$$

Valitaan $z = 3$

$$(2) \rightarrow y = 2 \cdot 3 = 6$$

$$(1) \rightarrow 3x - 3 \cdot 6 = -4 \cdot 3 \quad | :3 \\ x - 6 = -4 \\ x = 2$$

$$V: \begin{cases} x = 2 \\ y = 6 \\ z = 3 \end{cases}$$