CONDORCET'S JURY THEOREM UNDER RANDOM DECISIONAL POWER DISTRIBUTION

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ABSTRACT

Any organization, firm, or society follows a certain decision-making scheme. Here we are concerned with investigating the effectiveness of various weighted majority rules when more competent experts are not necessarily assigned more decisional power than less competent ones. Suppose we are given a vector of weights, and randomly distribute them among the experts. Is the average success probability when using any given rule lower than the success probability when using the simple majority rule? Under certain assumptions, we are able to show that the answer is positive for weighted majority rules, as well as some more general families of rules. This constitutes an interesting extension of CJT and may find applications in various areas.

1. INTRODUCTION

Issues related to democracy raise actual questions in many areas of society. A lot of works concerned with public choice theory, social science, mathematics and management explore various aspects of these questions. The theoretical origins of these studies goes back more than two centuries ago, as far as Condorcet [8]. His work plays a key role in the juncture of decision making and philosophy. Condorcet discovered two striking results about majority-rule voting. One is known as Condorcet's Paradox, demonstrating that majority voting is intransitive in general (cf. Elsholtz and List [10]). The second is known as "Condorcet's Jury Theorem" (henceforth CJT), and deals with some properties of majority voting in the dichotomous case (cf. Ben-Yashar and Paroush [1], Berend and Paroush [3]). Condorcet considered the following situation: a committee of $n$ experts is required to select one of two alternatives, of which exactly one is correct. For example, a committee of jurors has to decide whether a certain person is guilty. Condorcet believed that a group of individuals, facing a
binary choice and utilizing the simple majority rule, is more likely to make the correct choice than is a single decision maker. Moreover, this likelihood tends to complete certainty as the number of members of the group tends to infinity. A CJT is a formulation of conditions substantiating this belief. The classical conditions assume the independence of the decision makers and the same value $p > 0.5$ of the individual correctness probabilities. This statement provides the theoretical justification of democratic voting in public affairs and social choice (cf. McLean and Hewitt [13], Nurmi [15], Urken [17]). In practice, the classical assumptions of CJT, namely the homogeneity of the committee and independence among the experts, are seldom realistic in most practical situations. Thus, various attempts to generalize the theorem were made in several directions (cf. Berend and Paroush [3], Berg [5], Dietrich and List [9], Grofman et al. [11], Ladha [12], Nitzan and Paroush [14], Sapir [16]).

A decision rule is a rule for translating the individual opinions into a group decision. The most well-known group decision rule is the simple majority rule, strictly defined only for odd-sized committees. However, even for a committee of independent jurors, the simple majority rule is not necessarily the best procedure to follow, when the jurors have different decision skills. If the alternatives are symmetric, the experts are independent and the probability of each committee member to make the correct choice is known, Nitzan and Paroush [14] found a criterion for the identification of the optimal decision rule. In this case, the optimal decision rule is a member of the family of weighted majority rules (Nitzan and Paroush [14]).

However, there are situations in which decision power cannot be assigned according to expert competence levels, due to certain institutional constraints or simply since the competence levels of the experts are unknown. For example, it may be the case that a company manager, in spite of his lack of experience, is given more decisional power than some of his more experienced subordinates. Here we are concerned with investigating the effectiveness of various weighted majority rules when more competent experts are not necessarily assigned more weight than less competent ones. Suppose we are given a vector of weights, and randomly distribute the weights among the experts. In the spirit of Condorcet’s ideas, is the average success probability, using any given rule, lower than the success probability when using the simple majority rule? A full answer would provide an interesting extension of CJT and may find applications in various areas.

The rest of this paper is organized as follows: In Section 2 we introduce our formal model and notations. Section 3 presents the results obtained for the class of weighted majority rules,
attempting to clarify the connection between the "democracy level" of a rule and its probability of making the correct choice. In the sequel we present an extension CJT to broader class of rules.

2. THE MODEL

Let \{1,2,...,n\} be a committee of \(n\) experts, and denote by \(p_i, 1\leq i \leq n\), the probability of each expert \(i\) to make the right choice. We assume that the \(p_i\)'s are at least \(1/2\), and that the experts are independent in their choices. Given the experts' opinions, the final decision is taken by applying a decision rule, namely a rule for translating the individual opinions into a group decision. Let the random variable \(X_i\) be the choice of the \(i\)-th expert, where \(X_i = 1\) if the \(i\)-th expert chooses the first alternative and \(-1\) otherwise. In other words, a group decision rule is a function \(f\) from \((-1,1)^n\) to \((-1,1)\), which maps each decision profile \((x_1,x_2,...,x_n)\) where \(x_i\) represents the choice of the \(i\)-th expert, \(1\leq i \leq n\), to a collective decision. A decision rule \(f\) is neutral if \(f(-x_1,-x_2,...,-x_n) = -f(x_1,x_2,...,x_n)\) for every decision profile \((x_1,x_2,...,x_n)\). A decision rule \(f\) is monotonic if, for any decision profiles \((x_1,x_2,...,x_n), (\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n)\), such that \(x_i \geq \tilde{x}_i, 1\leq i \leq n\), we have \(f(x_1,x_2,...,x_n) = f(\tilde{x}_1,\tilde{x}_2,...,\tilde{x}_n)\). Clearly, the simple majority rule is monotonic and neutral, while a super majority rule, which typically needs a majority of \(2/3\) of the committee members in order to select the first alternative, is monotonic, but not neutral. Perhaps the most interesting class of monotonic and neutral rules is that of weighted majority rules, which, as shown by Nitzan and Paroush [14], contains the optimal decision rule in case of symmetric alternatives. A weighted majority rule assigns to each expert a weight, and the alternative chosen by a subgroup of experts with more than half the total weight is selected. The simple majority rule, strictly defined for odd \(n\), and the expert rule are weighted majority rules, defined by the systems of weights \((1,1,...,1)\) and \((1,0,...,0)\), respectively. However, the class of monotonic and neutral decision rules is much wider than that of weighted majority rules.

For a group of experts with competence structure \(\mathbf{p} = (p_1,p_2,...,p_n)\), and a system of weights \(\mathbf{w} = (w_1,w_2,...,w_n)\), denote by \(\prod \mathbf{w},\mathbf{p}\) the probability of choosing the correct
alternative when employing the weighted majority, the weights being from \( w \), distributed randomly among the experts.

For \( 1 \leq i \leq n \), denote by \( \rho_i(p) \) the probability that exactly \( i \) of the committee members choose the correct alternative, and by \( \gamma_i(w) \) the probability that the sum of \( i \) randomly selected \( w_j \)'s exceeds \( 1/2 \sum_{j=1}^{n} w_j \). With these notations we clearly have

\[
\prod_{w,p} = \sum_{i=1}^{n} \gamma_i(w) \rho_i(p).
\]

For any pair of decision rules \( f \) and \( g \), the rule \( f \) is (weakly) more effective than \( g \) if, for every competence structure of the group (still with \( p_i \geq 0.5 \), \( 1 \leq i \leq n \)), the probability of choosing the correct alternative by utilizing \( f \) when the weights are distributed randomly between the various committee members is (equal or) larger than the probability of choosing the correct alternative by utilizing \( g \) when the weights are distributed likewise.

3. EXTENSIONS OF CJT

In this section we study the connection between the effectiveness of various decision rules and their "democracy level". We deal first with weighted majority rules, and then with the more general class of neutral monotonic rules.

3.1. An Extension of CJT to Weighted Majority Rules

The first result in this direction is due to Ben-Yashar and Paroush [1], who compared the simple majority rule and expert rule utilized in a random way. Namely, they proved that the probability of a committee, utilizing the simple majority rule, to make a correct choice is larger than the probability \( \frac{1}{n} \sum_{i=1}^{n} p_i \) of a random single member to do so. Notice that the simple majority rule is invariant under permutations of the experts, but for the expert rule this permutation means that we choose a random committee member and follow his opinion. In terms of our definition of effectiveness, their result means that the simple majority rule is more effective than the expert rule. This implies that the probability of a random subgroup of any odd size \( k \leq n \) to reach the correct choice is larger than that of a random single member (Ben-Yashar and Paroush [1]). Later, Berend and Sapir [4] generalized this result and showed the monotonicity of the probability of a correct choice as a function of the size of the selected
subcommittee. In other words, they showed that the restricted majority rules become more effective as the number of influential committee members becomes larger. More formally, they proved the following result:

For a committee of independent experts of odd size \( n \geq 3 \) with arbitrary competence structure and odd \( 3 \leq k \leq n \), \( RMR_{n,k} \) is more effective than \( RMR_{n,k-2} \).

This raises the following questions:

- Is the simple majority rule the most effective of all weighted majority rules?
- Does the effectiveness increase as we get "closer" to the simple majority rule?

Recently, a positive answer to the first question was obtained by Berend and Chernyavsky [2], who generalized the results of Berend and Sapir [4] to the set of all weighted majority rules with randomly distributed power. Namely, they defined a rule as democratic if it decides according to the majority vote whenever there is such a majority. Thus, for odd \( n \), the only democratic rule is the simple majority rule, but for even \( n \) there are numerous democratic rules. For example, if \( n=4 \), then both rules \((2,1,1,1)\) and \((1,1,1,0)\) are democratic, while for \( n=6 \) all the rules \((1,1,1,1,1,0)\), \((2,1,1,1,1,1)\), \((2,2,2,1,1,1)\), \((3,3,2,2,2,1)\), \((3,2,2,2,1,1)\) and \((4,3,3,2,2,1)\) are such. Note that a rule is democratic if and only if it has no winning coalitions of size up to \( \left\lfloor \frac{n-1}{2} \right\rfloor \). Berend and Chernyavsky [2] obtained the following results:

- All the democratic rules are equally effective, and are strictly more effective than all other weighted majority rules.
- The expert rule is strictly less effective than all other weighted majority rules.

Hence, for odd \( n \), the simple majority rule is the most effective of all weighted majority rules. For even \( n \), the democratic rules, such as the restricted majority rule \( RMR_{n,n-1} = (1,\ldots,1,0) \) and chairman rule \( CH_n = (2,1,1,\ldots,1) \), are strictly more effective than all non-democratic rules.

However, no universally accepted definition of "democracy" exists (cf. [18]). Intuitively, sharing the decisional power evenly among all committee members, we obtain the ultimate level of democracy possible (represented, for committees of odd size, by the simple majority rule). On the other hand, a comparison of arbitrary weighted majority rules by their democracy levels is problematic in general and requires more study. Clearly, the rule represented by \((1,1,1,1,1,0,0)\) is more democratic than that of \((1,1,1,0,0,0,0)\). But what about
the two non-democratic rules $R_1 = (6,4,4,1,1,1,0)$ and $R_2 = (5,4,4,1,1,1,1)$? The same line of thought suggests that $R_2$ be considered more democratic, since all experts share in the decision, and the most powerful expert receives less power than in $R_1$. However, considering the percentage of winning coalitions of size up to $\lfloor (n-1)/2 \rfloor = 3$, we reach the following observation. Both rules have the same number of winning coalitions of size 1 and 2, but $R_2$ has 13 winning coalitions of size 3, whereas $R_1$ has only 12 such coalitions, making it, in this sense, closer to the democratic rules (Berend and Chernyavsky [2]).

The following theorem provides a sufficient condition for a weighted majority rule to be more effective than another.

*For a group of experts of size $n$ and two weighted majority rules $w, v$, if $\gamma_i(w) \leq \gamma_i(v)$ for $1 \leq i \leq \lfloor (n-1)/2 \rfloor$, then $w$ is more effective than $v$. If, moreover, $\gamma_i(w) < \gamma_i(v)$ for $1 \leq i \leq \lfloor (n-1)/2 \rfloor$, then $w$ is strictly more effective than $v$."

The theorem supports the belief that, the more democratic a rule is, the more effective it is.

Ideally, we would like to rank all the weighted majority rules by their effectiveness. However, it turns out that the situation is much more complex. Namely, Berend and Chernyavsky [2] proved that, for committees of $3 \leq n \leq 6$ experts, effectiveness provides a total (quasi-) ordering on the family of weighted majority rules, but this is not true anymore for $n \geq 7$.

However, if we restrict ourselves to certain families of rules in which the democracy level, in the sense mentioned previously, changes monotonically, effectiveness provides a total ordering on the rules of that family. For example, the main result of Berend and Sapir [4] shows that the single parameter $k$ provides a ranking of the rules in the class $\{RMR_{n,k} | 1 \leq k \leq n \}$ by their democracy level. Namely, by varying $k$ from 1 to $n$, we move from the least democratic rule to the most democratic one. Similarly, the same property holds for the family of distinguished expert rules $\{DC_{n,k} | 1 \leq k \leq n \}$ with $n, k$ of the same parity, which also connects the polar democratic rules (Berend and Chernyavsky [2]).
3.2. An Extension of CJT for Neutral and Monotonic Rules

In real-world situations, there are voting schemes other than WMR's. The indirect voting systems, which are common in various social and political structures (cf. Berg and Paroush[6], Boland [7]), are classical examples. Consider the indirect voting scheme in which there are 5 subcommittees of 3 experts each, where no expert serves in more than one subcommittee. It is not a member of WMR. However, it is monotonic and neutral.

The following results (unpublished yet) hold for wider classes of decision rules:

- The simple majority rule is strictly more effective than all other neutral decision rules.
- The expert rule is strictly less effective than all other monotonic neutral decision rules.

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