Integral Transforms (2018)

Exercise 3/Week 45

1. a) Calculate the Fourier series of the function

$$f(x) = x^2 \quad (x \in \mathbb{R})$$

on the interval $[-\pi, \pi]$.
b) Find the sum of this Fourier series and sketch its graph on the interval $[-3\pi, 3\pi]$.
c) Sketch the graphs of the first three partial sums of this Fourier series together with the graph of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

2. Using the Fourier series calculated in the previous exercise show that

$$\text{a)} \ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \text{b)} \ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

3. Calculate the Fourier series of the functions

$$\text{a)} \ f(x) = |\sin x|, \quad \text{b)} \ g(x) = \frac{\sin x + |\sin x|}{2} \quad (-\pi \leq x \leq \pi).$$

4. Using the Fourier series of the function $f(x) = x^2$ (cf. Exercise 8.1 above) find the best approximation of the form $a + b\cos x + c\sin x$ for the function

$$g(x) = \frac{x^2 + \sin(x)}{2}$$

with respect to the norm generated by the inner product $(f, g) = \int_{-\pi}^{\pi} f(t)g(t)\,dt$ in $C(-\pi, \pi)$.

5. Calculate the complex Fourier series of the function

$$f(x) = x^2 \quad (x \in \mathbb{R})$$

on the interval $[-\pi, \pi]$.

6. Apply the Fourier series of the function $f(x) = x^2$ and the Parseval's identity to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$