Integral Transforms (2018)

Exercise 4/Week 46

1. With a straightforward calculation show that the Fourier transform of the function

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

is given by $F(w) = \frac{i}{w}\, e^{-iw} + \frac{1}{w^2}\, e^{-iw} - \frac{1}{w^2}$.

2. a) Derive the following formula for the Fourier transform concerning general linear time transforms:

$$\mathcal{F}(f(at+b); w) = \frac{1}{|a|} e^{ibw/a}\, \mathcal{F}\left(f(t); \frac{w}{a}\right). \qquad (a, b \in \mathbb{R},\ a \neq 0)$$

b) Calculate the Fourier transform of the function

$$f(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

3. With a straightforward calculation show that the Fourier transform of the function

$$f(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad (a > 0)$$

is given by $F(w) = \frac{1}{a+iw}$.

4. By means of $\mathcal{F}$-transform of the function $f(t)$ in Exercise 4 and the Parseval's identity calculate the improper integral $\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}\, dw$.

5. Determine the amplitude spectrum and the phase spectrum of the function in the previous exercise. Sketch the graphs of these spectra, too.

6. Calculate the following improper integral

$$\int_{-\infty}^{\infty} \frac{te^{iwt}}{(t^2+a^2)^2}\, dt,$$

where $a > 0, w > 0$ (i.e. the $\mathcal{F}$-transform $F(-w)$ of $f(t) = \frac{t}{(t^2+a^2)^2}$) using
a) the formula for $\mathcal{F}$-transform of derivative and $\mathcal{F}$-transform tables;
b) the residue method.