Integral Transformations

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Definition

Let $\{V, (\cdot,\cdot)\}$ be an inner product space and let $\{f_1, \dots, f_n\}$ be an orthogonal sequence in V whose span is the subspace F_n. Then the projection onto F_n is defined as

$$P_{F_n}x = \sum_{i=1}^{n} \frac{(x, f_i)f_i}{||f_i||^2}, \quad x \in V.$$

In particular, if $\{e_1, \dots, e_n\}$ is an orthonormal sequence in V whose span is the subspace E_n, then the projection onto E_n is defined as

$$P_{E_n}x = \sum_{i=1}^{n} (x, e_i)e_i, \quad x \in V.$$

Proposition

Let $\{e_1, \dots, e_n\}$ be an orthonormal sequence in the inner product space $\{V, (\cdot,\cdot)\}$ and let E_n be its span. If $x, y \in E_n$, then

$$x = P_{E_n}x = \sum_{i=1}^{n} (x, e_i)e_i \quad \textit{and} \quad y = P_{E_n}y = \sum_{i=1}^{n} (y, e_i)e_i.$$

Moreover, Parseval's identity holds

$$(x, y) = \sum_{i=1}^{n} (x, e_i)\overline{(y, e_i)}.$$

Theorem

Let $\{e_1, \ldots, e_n\}$ *be an orthonormal sequence in the inner product space* $\{V, (\cdot, \cdot)\}$ *and let* E_n *be its span:*

$$E_n = \left\{ \sum_{i=1}^{n} a_i e_i : a_i \in \mathbb{C} \ or \ \mathbb{R} \right\}.$$

Moreover, let P_{E_n} *be as defined above. Then for any vector* $x \in V$*:*

$$||x - P_{E_n}x|| \leq ||x - y||, \quad \forall y \in E_n.$$

In other words, $P_{E_n}x$ is the element in E_n which is closest to x among all elements in E_n.

Approximation by trigonometric functions

Example

Recall that on the real vector space V_2,

$$V_2 = \{f : [-\pi, \pi] \to \mathbb{R} \text{ continuous} : \int_{-\pi}^{\pi} (f(s))^2 \mathrm{d}s < \infty\},$$

with the inner product $(\cdot, \cdot) : V_2 \times V_2 \to \mathbb{R}$ defined as

$$(f, g) = \int_{-\pi}^{\pi} f(s)g(s)\mathrm{d}s, \quad f, g \in V_2,$$

is an inner product and that for every $N \in \mathbb{N}$,

$$\{e_o, e_1, \ldots, e_{2N}\} := \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \ldots, \frac{\cos(Nx)}{\sqrt{\pi}}, \frac{\sin(Nx)}{\sqrt{\pi}} \right\}$$

is an orthonormal sequence in $\{V_2, (\cdot, \cdot)\}$. Now define E_N to be the span of the above e_i's:

$$E_N = \{a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) : a_0, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{R}\}.$$

Let f be any function in V_2, then the best approximation to this function in E_N is

$$\begin{aligned} P_{E_N} f &= (f, e_0)e_0 + \sum_{n=1}^N (f, e_{2n-1})e_{2n-1} + \sum_{n=1}^N (f, e_{2n})e_{2n} \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2\pi}} f(s) \right) \mathrm{d}s \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^N \int_{-\pi}^{\pi} \left(f(s) \frac{\cos(ns)}{\sqrt{\pi}} \right) \mathrm{d}s \frac{\cos(nx)}{\sqrt{\pi}} + \sum_{n=1}^N \int_{-\pi}^{\pi} \left(f(s) \frac{\sin(ns)}{\sqrt{\pi}} \right) \mathrm{d}s \frac{\sin(nx)}{\sqrt{\pi}} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\mathrm{d}s + \frac{1}{\pi} \sum_{n=1}^N \int_{-\pi}^{\pi} (f(s)\cos(ns))\,\mathrm{d}s \cos(nx) + \frac{1}{\pi} \sum_{n=1}^N \int_{-\pi}^{\pi} (f(s)\sin(ns))\,\mathrm{d}s \sin(nx). \end{aligned}$$

Now for a function f, define

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s)\mathrm{d}s, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s)\cos(ns)\mathrm{d}s, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s)\sin(ns)\mathrm{d}s.$$

Then

$$P_{E_N} f = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx)).$$

I.e., the partial Fourier series is the best approximation in the space of $\cos nx$ and $\sin nx$ functions up to a certain frequency. Therefore the Fourier series itself ("$N = \infty$") is the best approximation in the space of all $\cos nx$ and $\sin nx$ functions.

Recall the following useful calculation rules for Fourier series:

1. $S_{af+g}(x) = aS_f(x) + S_g(x)$, $a \in \mathbb{R}$;
2. if f is even, i.e. $f(-x) = f(x)$, then $b_n = 0$, $n = 1, \ldots$;
3. if f is odd, i.e. $f(-x) = -f(x)$, then $a_n = 0$, $n = 0, \ldots$;
4. if $f(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(nx) + b_n \sin(nx))$, then $S_f(x) = f(x)$.

Example

1. The Fourier series S_f of $f(x) = 2\cos(3x) + 10\sin(8x)$, $-\pi \leq x \leq \pi$ is f itself. Because the Fourier series is the best approximant in the space spanned by $1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots$ and f is already in that space.
2. Let $g(x) = x, -\pi \leq x \leq \pi$, determine its Fourier series and find the best approximant of f in the space spanned by $\{1, \cos(x), \sin(x)\}$ and by $\{1, \cos(x), \sin(x), \sin(2x), \cos(2x)\}$.

   The first step is to determine the Fourier series of g. Since g is an odd function, only the b_n's need to be determined:

$$\begin{aligned} b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)dt = \tfrac{1}{\pi}\left[\left[-t\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nt)}{n}\mathrm{d}t\right] \\ &= \tfrac{1}{n\pi}\left[-\pi(-1)^n - \pi(-1)^n\right] + \tfrac{1}{\pi n^2}\left[\sin(nt)\right]_{-\pi}^{\pi} = -\frac{2(-1)^n}{n} \end{aligned}$$

   Hence, the Fourier series of g is given by

$$S_g(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) = 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \ldots\right)$$

   and the asked for approximants are $2\sin(x)$ and $2\sin(x) - \sin(2x)$, respectively.
3. Determine the Fourier series of $h(x) = x + 2\cos(3x) + 10\sin(8x)$, $-\pi \leq x \leq \pi$.
   This problem can be solve by using the linearity property of Fourier series:

$$S_h(x) = S_f(x) + S_g(x) = 2\cos(3x) + 10\sin(8x) + 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \ldots\right).$$

Complex Fourier series

Fourier series can also be written with respect to an exponential basis. Therefore remember:

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i} \quad \text{and} \quad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}.$$

With the help of these formulas one has that

$$a_n \cos(nx) + b_n \sin(nx) = a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} = \frac{a_n - ib_n}{2} e^{inx} + \frac{a_n + ib_n}{2} e^{-inx}.$$

Now define the coefficient of e^{inx} to be c_n, i.e

$$c_n = \frac{a_n - ib_n}{2} \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n \geq 0,$$

where $c_0 = a_0/2$, because $b_0 = 0$. Note that $\overline{c_n} = c_{-n}$ for $n \in \mathbb{Z}$ and that

$$a_n = 2\mathrm{Re}\, c_n \quad \text{and} \quad b_n = -2\mathrm{Im}\, c_n, \quad n \geq 0.$$

In this way one has obtained the complex representation of the Fourier series S_f for a function f:

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} := \lim_{m\to\infty} \sum_{n=-m}^{m} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \mathrm{d}x.$$

Let f be a 2π-periodic function and let c_n be its complex Fourier coefficients. Then

1. the set $\{(n, c_n) : n \in \mathbb{Z}\}$ is called its *spectrum*;
2. the set $\{(n, |c_n|) : n \in \mathbb{Z}\}$ is called its *amplitude spectrum*;
3. the set $\{(n, \arg(c_n)) : n \in \mathbb{Z}\}$ is called its *phase spectrum*.

The spectrum gives information about the harmonic frequency components e^{inx}; the amplitude spectrum gives the strength of the harmonic frequency under consideration and the phase spectrum the phase in which the frequency starts.

Proposition

(Parseval's identity) Let f be a function and let c_n be its complex Fourier coefficients and a_n and b_n its real Fourier coefficients. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}.$$

Example

Recall that if $f(x) = x, -\pi \leq x \leq \pi$, then

$$S_f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \dots\right).$$

Now $c_0 = a_0/2 = 0$,

$$c_n = \frac{a_n - ib_n}{2} = \frac{-2i\frac{(-1)^{n+1}}{n}}{2} = i\frac{(-1)^n}{n}, \quad n > 0,$$

and

$$c_{-n} = \overline{c_n} = -i\frac{(-1)^n}{n} = i\frac{(-1)^{-n}}{-n}, \quad n > 0.$$

Hence, $f(x)$ has the following complex Fourier series representation:

$$S_f(x) = i\sum_{n=-\infty, n\neq 0}^{\infty} \frac{(-1)^n}{n} e^{inx}.$$

The coefficients could also have been calculated directly. For $n = 0$:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\mathrm{d}x = \left[\frac{x^2}{2\pi}\right]_{-\pi}^{\pi} = 0.$$

and for $n \in \mathbb{Z} \setminus \{0\}$:

$$\begin{aligned} c_n &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\mathrm{d}x = \tfrac{1}{2\pi}\left(\left[x\tfrac{e^{-inx}}{-in}\right]_{-\pi}^{\pi} + \tfrac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\mathrm{d}x\right) \\ &= \tfrac{1}{2\pi}\left(\pi\tfrac{e^{-i\pi n}}{-in} + \pi\tfrac{e^{i\pi n}}{-in} + 0\right) \\ &= \tfrac{1}{-in}\left(\tfrac{e^{i\pi n}+e^{-i\pi n}}{2}\right) = \tfrac{i}{n}\cos(n\pi) = \tfrac{i}{n}(-1)^n. \end{aligned}$$

The amplitude and phase spectrum of the function are given by

$$\begin{aligned}
\text{amplitude spectrum:} \quad \{(n, |c_n|) : n \in \mathbb{Z}\} &= \{(0,0)\} \cup \{(n, \left|i\tfrac{(-1)^n}{n}\right|) : n \in \mathbb{Z} \setminus \{0\}\} \\
&= \{(0,0)\} \cup \{(n, 1/n) : n \in \mathbb{Z} \setminus \{0\}\}; \\
\text{phase spectrum:} \quad \{(n, \arg(c_n)) : n \in \mathbb{Z}\} &= \{(n, \arg(i\tfrac{(-1)^n}{n})) : n \in \mathbb{Z} \setminus \{0\}\} \\
&= \{(n, \arg(\operatorname{sgn}(n) i (-1)^n)) : n \in \mathbb{Z} \setminus \{0\}\} \\
&= \{(n, \operatorname{sgn}(n)(-1)^n \tfrac{\pi}{2}) : n \in \mathbb{Z} \setminus \{0\}\}.
\end{aligned}$$

Figure: The amplitude spectrum.

Figure: The phase spectrum.

Furthermore, Parseval's identity yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty, n\neq 0}^{\infty} \left|\frac{(-1)^n}{n}\right|^2.$$

Now

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

and

$$\sum_{n=-\infty, n\neq 0}^{\infty} \left|\frac{(-1)^n}{n}\right|^2 = \sum_{n=-\infty, n\neq 0}^{\infty} \left|\frac{1}{n}\right|^2 = 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore Parseval's identity yields:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\frac{\pi^2}{3} = \frac{\pi^2}{6}.$$

Fourier series on arbitrary intervals

For a $2L$-periodic function, or for a function defined on $[-L, L]$, a Fourier series can be defined:

$$S_f^L(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

This Fourier series can also be written in complex form:

$$S_f^L(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi}{L}x}.$$

In this series the frequencies and angular frequencies of the functions are

$$\frac{n}{2L} \quad \text{and} \quad n\frac{\pi}{L}, \quad n \in \mathbb{N},$$

whereas in a "normal" Fourier series the frequencies and angular frequencies of the functions are

$$\frac{n}{2\pi} \quad \text{and} \quad n, \quad n \in \mathbb{N}.$$

Example

Find the $2L$-periodic, $L > 1$, Fourier series of the function

$$f(x) = \begin{cases} 1, & -1 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since the function is even, only the a_n-coefficients need to be calculated:

$$\begin{aligned} a_0 &= \tfrac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x = \tfrac{1}{L}\int_{-1}^{1} 1\mathrm{d}x = \tfrac{2}{L}; \\ a_n &= \tfrac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x = \tfrac{1}{L}\int_{-1}^{1}\cos\left(\tfrac{n\pi x}{L}\right)\mathrm{d}x = \tfrac{1}{L}\left[\tfrac{L}{n\pi}\sin\left(\tfrac{n\pi x}{L}\right)\right]_{-1}^{1} \\ &= \tfrac{1}{n\pi}\left(\sin\left(\tfrac{n\pi}{L}\right) - \sin\left(-\tfrac{n\pi}{L}\right)\right) = \tfrac{2}{L}\tfrac{\sin(n\pi/L)}{n\pi/L}. \end{aligned}$$

Hence, the function's $2L$-periodic Fourier series is

$$S_f^L(x) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin(n\pi/L)}{n\pi/L}\cos\left(\frac{n\pi}{L}x\right).$$

Fourier transformation

Let $f(x)$ be a function which on each interval, around 0, of length $2L$ can be represented by a Fourier series. Then

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(w_n x) + \sin(w_n x)), \quad w_n = \frac{n\pi}{L}.$$

Plugging in the definition of the a_i and b_i yields

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{L}\sum_{n=1}^{\infty}\left(\cos(w_n x)\int_{-L}^{L} f(s)\cos(w_n s)\mathrm{d}s + \sin(w_n x)\int_{-L}^{L} f(s)\sin(w_n s)\mathrm{d}s\right).$$

Let Δw be the difference between the different (angular) frequencies:

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $f(x)$ can be written as:

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(s)\mathrm{d}s + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos(w_n x)\int_{-L}^{L} f(s)\cos(w_n s)\mathrm{d}s + \sin(w_n x)\int_{-L}^{L} f(s)\sin(w_n s)\mathrm{d}s\right)\Delta w.$$

Assuming f to be absolutely integrable, sending L to ∞ gives us the *Fourier integral*

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\cos(wx)\int_{-\infty}^{\infty} f(s)\cos(ws)\mathrm{d}s + \sin(wx)\int_{-\infty}^{\infty} f(s)\sin(ws)\mathrm{d}s\right)\mathrm{d}w.$$

The Fourier integral can also be written in complex form:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\left(\int_{-\infty}^{\infty} e^{-iws} f(s)\mathrm{d}s\right)\mathrm{d}w.$$

Based on this result, the concept of a Fourier transform is introduced: For an absolutely integrable function f the Fourier transform of f is the function

$$\mathcal{F}(f;w) = \int_{-\infty}^{\infty} e^{-iws} f(s)\mathrm{d}s,$$

which is also denoted as $F(w)$. The inverse Fourier transform of $F(w)$ is

$$f(x) = \mathcal{F}^{-1}(F(w);x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx} F(w)\mathrm{d}w.$$

Remark

Based on our deduction, the following interpretation for the Fourier transform $F(w)$ of a function $f(x)$ has been obtained: $F(w)$ measures the intensity of $f(x)$ in the (angular) frequency interval between w and $w + \Delta w$.

The Fourier transform: Definition

Let f be a real-valued or complex-valued function, then its Fourier transform $\mathcal{F}(f(t); w)$ also denoted by $\mathcal{F}_f(w)$ or $F(w)$, is defined as

$$\mathcal{F}(f(t); w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\mathrm{d}t, \quad w \in \mathbb{R}.$$

This transformation is well defined for all $w \in \mathbb{R}$ if f is absolutely integrable over $\mathbb{R}$:

$$\int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < \infty.$$

The inverse Fourier transform of a function g, which is absolutely integrable, is defined as

$$\mathcal{F}_g^{-1}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(w)e^{iwt}\mathrm{d}w.$$

The Fourier and the inverse Fourier transforms are each others inverses:

$$\mathcal{F}_{\mathcal{F}_f}^{-1}(t) = f(t) \quad \text{and} \quad \mathcal{F}_{\mathcal{F}_g^{-1}}(w) = g(w).$$

The functions

$$|F(w)| \quad \text{and} \quad \arg F(w)$$

are called the *amplitude* and *phase spectrum function* of f, respectively.

Example

Let f be

$$f(t) = \begin{cases} k, & -a < t < a; \\ 0, & otherwise, \end{cases}$$

where $a > 0$. Then

$$\begin{aligned} \mathcal{F}_f(w) &= \int_{-\infty}^{\infty} f(t)e^{-iwt}\mathrm{d}t = \int_{-a}^{a} ke^{-iwt}\mathrm{d}t = k\left[\frac{1}{-iw}e^{-iwt}\right]_{-a}^{a} \\ &= k\left(\frac{1}{-iw}e^{-iaw} + \frac{1}{iw}e^{iaw}\right) = \frac{2k}{w}\frac{e^{iaw}-e^{-iaw}}{2i} = \frac{2k\sin(aw)}{w}. \end{aligned}$$

Example

Let g be

$$g(t) = e^{-a|t|}, \quad a > 0.$$

Then

$$\begin{aligned} G(w) &= \int_{-\infty}^{\infty} g(t)e^{-iwt}\mathrm{d}t = \int_{-\infty}^{0} e^{at}e^{-iwt}\mathrm{d}t + \int_{0}^{\infty} e^{-at}e^{-iwt}\mathrm{d}t \\ &= \int_{-\infty}^{0} e^{(a-iw)t}\mathrm{d}t + \int_{0}^{\infty} e^{-(a+iw)t}\mathrm{d}t = \left[\frac{e^{(a-iw)t}}{a-iw}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+iw)t}}{-(a+iw)}\right]_{0}^{\infty} \\ &= \frac{1}{a-iw} - \frac{1}{-(a+iw)} = \frac{2a}{a^2+w^2}. \end{aligned}$$

The Fourier transform: Basic properties

Proposition

Let f and g be absolutely integrable and let $a, b \in \mathbb{C}$, then

1. $\mathcal{F}(af(t) + bg(t); w) = a\mathcal{F}(f(t); w) + b\mathcal{F}(g(t); w)$;
2. $\mathcal{F}(f(at); w) = \frac{1}{a}\mathcal{F}(f(t); \frac{w}{a}), \quad a > 0$;
3. $\mathcal{F}(f(t - t_0); w) = e^{-iwt_0}\mathcal{F}(f(t); w)$;
4. $\mathcal{F}(f(t)e^{iw_0 t}; w) = \mathcal{F}(f(t); w - w_0)$.

The Fourier transformation also satisfies Parseval's identity:

Proposition

(Parseval) Let f be an absolutely integrable function. Then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F}_f(w)|^2 \, dw.$$

Example

Calculate the Fourier transform of f, where

$$f(t) = \begin{cases} -2, & -2\pi < t < -\pi; \\ 2, & \pi < t < 2\pi; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f(-t) = -f(t)$. Define $f_1(t)$ and $f_2(t)$ as

$$f_1(t) = \begin{cases} -2, & -2\pi < t < -\pi; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 2, & \pi < t < 2\pi; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = f_1 + f_2$.

Define g as

$$g(t) = \begin{cases} 1, & -\pi/2 < t < \pi/2; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f(t) = f_1(t) + f_2(t) = -2g(t + 3\pi/2) + 2g(t - 3\pi/2)$. Hence, using the result of a previous exercise and the above proposition yields

$$\begin{aligned} \mathcal{F}(f(t); w) &= \mathcal{F}(-2g(t + 3\pi/2) + 2g(t - 3\pi/2); w) \\ &= -2\mathcal{F}(g(t + 3\pi/2); w) + 2\mathcal{F}(g(t - 3\pi/2); w) \\ &= -2e^{3\pi iw/2}\mathcal{F}(g(t); w) + 2e^{-3\pi wi/2}\mathcal{F}(g(t); w) \\ &= -2(e^{3\pi iw/2} - e^{-3\pi wi/2})\mathcal{F}(g(t); w) \\ &= -4i\,\frac{e^{3\pi iw/2} - e^{-3\pi wi/2}}{2i}\,\frac{2\sin(\pi w/2)}{w} \\ &= \frac{-8i}{w}\sin(3\pi w/2)\sin(\pi w/2). \end{aligned}$$

Example

Let f be given by

$$f(t) = \begin{cases} 1, & -1 < t < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}_f(w) = 2\frac{\sin(w)}{w}$ and a direct calculation shows that

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = \int_{-1}^{1} 1 \mathrm{d}t = 2,$$

Furthermore,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}_f(w)|^2 \mathrm{d}w = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2\sin(w)}{w}\right)^2 \mathrm{d}w = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin(w)}{w}\right)^2 \mathrm{d}w.$$

Hence, Parseval's identity yields

$$\int_{-\infty}^{\infty} \left(\frac{\sin(w)}{w}\right)^2 \mathrm{d}w = 2\frac{\pi}{2} = \pi.$$

The Fourier transform: Transform of the δ-function

The delta function $\delta(t)$ is formally defined by means of integrals:

$$\int_a^b \delta(t-t_0)f(t)\mathrm{d}t = \begin{cases} f(t_0), & a \le t_0 \le b; \\ 0, & \textit{otherwise}. \end{cases}$$

In particular,

$$\int_a^b \delta(t)\mathrm{d}t = \begin{cases} 1, & a \le 0 \le b; \\ 0, & \textit{otherwise}. \end{cases}$$

The Fourier transform of the δ function is easily determined:

$$\mathcal{F}(\delta; w) = \int_{-\infty}^{\infty} \delta(t)e^{-iwt}\mathrm{d}t = e^{-iw0} = 1.$$

A similar calculation shows that $\mathcal{F}^{-1}(\delta; t) = (2\pi)^{-1}$. Therefore $\mathcal{F}(1/2\pi; w) = \delta(w)$ or, equivalently,

$$\mathcal{F}(1; w) = 2\pi\delta(w).$$

Using the shifting property of Fourier transform, the preceding result yields

$$\mathcal{F}(e^{iw_0t}; w) = \mathcal{F}(1 \cdot e^{iw_0t}; w) = \mathcal{F}(1; w-w_0) = 2\pi\delta(w-w_0).$$

In particular,

$$\mathcal{F}\left(\sum_{k=1}^{n} a_k e^{iw_kt}; w\right) = \sum_{k=1}^{n} a_k \mathcal{F}(e^{iw_kt}; w) = 2\pi \sum_{k=1}^{n} a_k \delta(w-w_k).$$

This shows that the Fourier transform maps the oscillations e^{-iw_kt} "to their corresponding frequencies" w_k.

The Fourier transform: Differentiation

Proposition

Assume that f has a derivative and that $|f|$ and $|f'|$ are absolutely integrable over $\mathbb{R}$. Then

$$\mathcal{F}(f'(t); w) = iw\mathcal{F}(f(t); w).$$

In particular, if f is n times differentiable and $|f|$ and all the derivatives $|f'|$, $|f^{(2)}|$, ... $|f^{(n)}|$ are absolutely integrable, then

$$\mathcal{F}(f^{(n)}(t); w) = (iw)^n\mathcal{F}(f(t); w).$$

Proposition

Let f be absolutely integrable and piecewise smooth and if $t^m f(t)$, $m \in \mathbb{N}$, has a Fourier transform, then

$$\mathcal{F}(t^m f(t); w) = i^m \frac{d^m}{dw^m}\mathcal{F}(f(t); w).$$

Example

Determine the Fourier transform of

$$f(t) = te^{-t^2/2}.$$

By definition

$$\mathcal{F}(f(t); w) = \mathcal{F}\left(te^{-t^2/2}; w\right) = -\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{-t^2/2}; w\right) = -iw\mathcal{F}\left(e^{-t^2/2}, w\right).$$

From the tables you can find $\mathcal{F}\left(e^{-t^2/2}; w\right) = \sqrt{2\pi}e^{-w^2/2}$. Consequently,

$$\mathcal{F}(f(t), w) = -\sqrt{2\pi}iwe^{-w^2/2}.$$

Example

Determine the Fourier transform of

$$f(t) = u(t)te^{-at} = \begin{cases} te^{-at}, & t > 0; \\ 0, & t < 0 \end{cases}, \quad a > 0.$$

Using the second rule

$$\mathcal{F}(f(t); w) = \mathcal{F}(tu(t)e^{-at}; w) = i\frac{\mathrm{d}}{\mathrm{d}w}\mathcal{F}(u(t)e^{-at}; w) = i\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{a + iw} = \frac{1}{(a + iw)^2}.$$

Here the fact that $\mathcal{F}(u(t)e^{-at}; w) = (a + iw)^{-1}$, which can be found from the tables, was used.

The Fourier transform: Convolution

Let f and g be functions defined on $\mathbb{R}$, then the convolution of f and g is the function $f * g$ which is defined by

$$(f * g)(t) := \int_{-\infty}^{\infty} f(s)g(t-s)\mathrm{d}s = \int_{-\infty}^{\infty} f(t-s)g(s)\mathrm{d}s = (g * f)(t).$$

The convolution can be used to calculate the product of Fourier transforms.

Proposition

Let f and g be continuous and absolutely integrable on $\mathbb{R}$. Then

$$\mathcal{F}((f * g)(t); w) = \mathcal{F}(f(t); w)\mathcal{F}(g(t); w).$$

Example

Let $f(t) = e^{-a|t|}$ and let $g(t) = \cos(at)$, where $a > 0$, then determine their convolution. By the preceding statement

$$\mathcal{F}(f * g; w) = \mathcal{F}(f(t); w) \cdot \mathcal{F}(g(t); w) = \frac{2a}{w^2 + a^2} \cdot \pi(\delta(w-a) + \delta(w+a)),$$

where the tables were used. Taking the inverse Fourier transform on both sides yields:

$$\begin{aligned}(f * g)(t) &= \mathcal{F}^{-1}\left(\tfrac{2a}{w^2+a^2} \cdot \pi(\delta(w-a) + \delta(w+a)); t\right)\\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{2a}{w^2+a^2} \cdot \pi(\delta(w-a) + \delta(w+a))e^{iwt}\mathrm{d}w\\ &= \int_{-\infty}^{\infty} \tfrac{a}{w^2+a^2} \cdot (\delta(w-a) + \delta(w+a))e^{iwt}\mathrm{d}w\\ &= \tfrac{a}{a^2+a^2}e^{iat} + \tfrac{a}{(-a)^2+a^2}e^{-iat} = \tfrac{1}{a}\tfrac{e^{iat}+e^{-iat}}{2} = \tfrac{\cos(at)}{a}.\end{aligned}$$

The Fourier transform: Application to differential equations

Example

For a fixed function r and for $a \notin \mathbb{N}$, solve the differential equation

$$-y''(t) + a^2 y(t) = r(t).$$

To solve this problem use the Fourier transform: First transform the lefthand side

$$\begin{aligned}\mathcal{F}(-y''(t) + a^2 y(t); w) &= -\mathcal{F}(y''(t); w) + a^2 \mathcal{F}(y(t); w) \\ &= -(iw)^2 \mathcal{F}(y, w) + a^2 \mathcal{F}(y, w) \\ &= (w^2 + a^2)\mathcal{F}(y, w).\end{aligned}$$

Hence by Fourier transforming our differential equation we obtain that

$$(w^2 + a^2)\mathcal{F}(y(t); w) = \mathcal{F}(r(t); w)$$

or, equivalently, our Fourier transform $\mathcal{F}(y(t); w)$ is given by

$$\mathcal{F}(y(t); w) = \frac{1}{w^2 + a^2}\mathcal{F}(r(t); w).$$

From the tables one finds that

$$\mathcal{F}(e^{-a|t|}; w) = \frac{2a}{w^2 + a^2}, \quad a > 0.$$

Hence (with $a > 0$)

$$\mathcal{F}(e^{-a\cdot|t|}; w) = \frac{2a}{w^2 + a^2}.$$

Combining this with our obtained expression for the Fourier transform of y yields that

$$\mathcal{F}(y(t); w) = \frac{1}{w^2 + a^2}\mathcal{F}(r; w) = \mathcal{F}\left(\frac{e^{-a\cdot|t|}}{2a}; w\right)\mathcal{F}(r(t); w) = \mathcal{F}\left(\left(\frac{e^{-a\cdot|u|}}{2a} * r(u)\right)(t); w\right).$$

Consequently,

$$y(t) = \frac{1}{2a}\left(e^{-a\cdot|u|} * r(u)\right)(t) = \frac{1}{2a}\int_{-\infty}^{\infty} e^{-a\cdot|t-u|} r(u) du.$$

The above solution is not an unique solution of the differential equation, because no initial conditions were given. Therefore solutions of the homogeneous differential equation

$$-y''(t) + a^2 y(t) = 0$$

can be added to the obtained solution.

In particular, if $r(t) = \delta(t)$, then the solution

$$y(t) = \frac{1}{2a}\left(e^{-a\cdot|u|} * r(u)\right)(t) = \frac{1}{2a}e^{-a\cdot|t|}.$$

of the differential equation $-y''(t) + a^2 y(t) = \delta(t)$ is found

Heat equation on an infinite line

The problem of the flow of heat in an infinite medium with initial temperature distribution $f(x)$ and heat source $q(x,t)$ can be mathematically modeled as follows:

$$\begin{array}{ll} & u_{xx}(x,t) = a^{-2}u_t(x,t) + q(x,t), \quad -\infty < x < \infty, t > 0, \\ \text{B.C.:} & \lim_{|x|\to\infty} u(x,t) = 0, \lim_{|x|\to\infty} u_x(x,t) = 0, \\ \text{I.C.:} & u(x,0) = f(x), \quad -\infty < x < 0. \end{array}$$

Here $u(x,t)$ is the heat at point x and time t, and $a > 0$ is a thermal diffusivity constant. For simplicity assume that $q(x,t) = 0$. To solve this problem take the Fourier transform of u with respect to the spacial variable:

$$\begin{aligned} 0 &= \mathcal{F}(u_{xx}(x,t) - a^{-2}u_t(x,t); x \to w) = \mathcal{F}(u_{xx}(x,t); x \to w) - a^{-2}\mathcal{F}(u_t(x,t); x \to w) \\ &= (iw)^2\mathcal{F}(u(x,t); x \to w) - a^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(x,t); x \to w). \end{aligned}$$

Denoting $\mathcal{F}(u(x,t); x \to w)$ by $U(w,t)$, the following initial value problem has been obtained:

$$a^2w^2U(w,t) + U_t(w,t) = 0, \quad U(w,0) = F(w) = \mathcal{F}(f(x); w).$$

This first-order differential equation can be solved:

$$U(w,t) = F(w)e^{-a^2w^2t} = \mathcal{F}(f(x); w) \cdot \mathcal{F}\left(\frac{e^{-\frac{x^2}{4a^2t}}}{a\sqrt{2\pi t}}; x \to w\right).$$

Hence, using the convolution theorem and taking inverse Fourier transforms yields

$$u(x,t) = \left(\frac{e^{-\frac{u^2}{4a^2t}}}{2a\sqrt{\pi t}} * f(u)\right)(x) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(s)e^{-\frac{(x-s)^2}{4a^2t}}\,\mathrm{d}s.$$

Infinite beam on resting on an elastic foundation

If a load $f(x)$ is placed on an infinite beam, then the deflection $y(x)$ of the beam should satisfy:

$$EIy^{(4)}(x) + ky(x) = f(x), \quad -\infty < x < \infty.$$

Here E, I and k are positive constants which all have a physical interpretation. Now consider the problem that there exists a constant $F_0 > 0$ such that the load f is given by

$$f(x) = \begin{cases} F_0, & -1 < x < 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Applying the Fourier transform to both sides gives, using the tables, that

$$\begin{aligned}\frac{2F_0\sin(w)}{w} &= \mathcal{F}(f(x);w) = \mathcal{F}(EIy^{(4)}(x)+ky(x);w) = EI(iw)^4\mathcal{F}(y(x);w)+k\mathcal{F}(y(x);w)\\ &= (EIw^4+k)\mathcal{F}(y(x);w).\end{aligned}$$

In other words,

$$y(x) = \mathcal{F}^{-1}\left(\frac{2F_0\sin(w)}{w(EIw^4+k)};x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2F_0\sin(w)}{w(EIw^4+k)}e^{iwx}\mathrm{d}w.$$

Because $2F_0\sin(w)/w(EIw^4+k)$ is an even function, the above result can be simplified to

$$y(x) = \frac{F_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin(w)}{w(EIw^4+k)}\cos(wx)\mathrm{d}w.$$

On the righthand side residue calculus can be used, therefore note that the only poles in the upper halfplane of the integrand are at $c\cdot e^{\pi/4}$ and $c\cdot e^{3\pi/4}$, where c is the positive fourth root out of EI/k. Using those residue's one obtains

$$y(x) = \frac{F_0}{2k}\left(e^{\frac{-c(1+x)}{\sqrt{2}}}\sin\frac{c(1+x)}{\sqrt{2}} + e^{\frac{-c(1-x)}{\sqrt{2}}}\sin\frac{c(1-x)}{\sqrt{2}}\right).$$

The $\mathcal{Z}$-transform

As discussed in the section on the discrete Fourier transform for practical purposes one rather works with discrete signals. Therefore next a discrete analogue of the Laplace transform, the $\mathcal{Z}$-transform, will be considered. This is a transform that converts a discrete signal into a complex frequency domain representation and is used in communication systems.

Example

A typical example of a difference problem is the following: Find a sequence $\{y(n)\}$ which satisfies

$$a \cdot y(n+2) + b \cdot y(n+1) + c \cdot y(n) = x(n), \quad n = 0, 1, 2, \dots,$$

where a, b and c are constants and $\{x(n)\}$ is a fixed (known) sequence. To have a unique solution one needs, like in differential equations, initial conditions, which could for instance take the form $y(0) = 0$ and $y(1) = 1$.

The $\mathcal{Z}$-transform: An introduction

Let $\{x(n)\}_{n\in\mathbb{Z}}$ be a (2-sided) infinite sequence of complex numbers, i.e.

$$\{x(n)\} = \{\dots, x(-2), x(-1), x(0), x(1), x(2), \dots\},$$

then its $\mathcal{Z}$-transform, denoted by $\mathcal{Z}(x(n); z)$ or $\mathcal{X}(z)$, is the (formal) expression

$$\mathcal{Z}(x(n); z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}. \tag{7.1}$$

I.e., the $\mathcal{Z}$-transform of a sequence is a Laurent series at 0, which therefore a well-defined function (converges) in an annulus or nowhere. In applications, one usually deals with *causal* sequences $\{x(n)\}$, which means that $x(n) = 0$ if $n < 0$, cf. the Laplace transform.

Since the $\mathcal{Z}$-transform is a Laurent series, the sequence $\{x(n)\}$ can be recovered from its $\mathcal{Z}$-transform:

$$x(n) = \frac{1}{2\pi i} \oint_C \frac{\mathcal{X}(z)}{z^{-n+1}} dz, \quad n \in \mathbb{Z},$$

where C is a closed curve contained in the annulus where $\mathcal{X}(z)$ converges. Alternatively, residue calculus might be used to calculate the coefficients $x(n)$ of a causal sequence:

$$x(n) = \sum_{a_i} \mathrm{Res}_{z=a_i} \left(\frac{\mathcal{X}(z)}{z^{-n+1}} \right), \quad n \in \mathbb{Z},$$

where the sum is over all the poles a_i of the function $\mathcal{X}(z)$.

Example

1. Consider the sequence $\{x(n)\}$, where $x(-2) = -4$, $x(0) = 1$, $x(1) = 10$, and all the other coefficient are zero. Then

$$\mathcal{Z}(x(n); z) = -4z^2 + 1 + 10z^{-1}.$$

2. Let $\{x(n)\}$ be a causal sequence, where $x(n) = a^n$ for $a \in \mathbb{C}$, $n \in \mathbb{N}$. Then

$$\mathcal{Z}(x(n); z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}.$$

   Here use of the geometrical series was made, from which it in particular follows that the preceding $\mathcal{Z}$-transform converges for (is well defined for) $|z| > |a|$.

3. Let $\{x(n)\}$ be a causal sequence, where $x(n) = n$, $n \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{Z}(x(n); z) &= \textstyle\sum_{n=0}^{\infty} nz^{-n} = z\sum_{n=0}^{\infty} nz^{-n-1} = -z\frac{d}{dz}\sum_{n=0}^{\infty} z^{-n} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) \\ &= -z\left(\frac{-1}{(z-1)^2}\right) = \frac{z}{(z-1)^2}.\end{aligned}$$

   Here the preceding $\mathcal{Z}$-transform converges for (is well defined for) $|z| > 1$.

The $\mathcal{Z}$-transform: Basic properties

$\mathcal{Z}$-transform of causal series have properties quite similar to those of the Laplace transform.

Proposition

Let $\{x(n)\}$ and $\{y(n)\}$ be causal series and let $\mathcal{X}(z)$ and $\mathcal{Y}(z)$ be their $\mathcal{Z}$-transforms which converge in $D_1 = \{\alpha < |z| < \beta\}$ and $D_2 = \{\gamma < |z| < \delta\}$, respectively, and let $a, b \in \mathbb{C}$. Then

1. $\mathcal{Z}(ax(n) + by(n); z) = a\mathcal{X}(z) + b\mathcal{Y}(z)$, *where* $D = D_1 \cap D_2$;
2. $\mathcal{Z}(x(n - n_0); z) = z^{-n_0}\mathcal{X}(z) + z^{-n_0}\sum_{m=-n_0}^{-1} x(m)z^{-m}$, *where* $D = D_1$ *and* $n_0 \geq 0$;
3. $\mathcal{Z}(x(n + n_0); z) = z^{n_0}\mathcal{X}(z) - \sum_{m=0}^{n_0-1} x(m)z^{n_0-m}$, *where* $D = D_1$ *and* $n_0 \geq 0$;
4. $\mathcal{Z}(a^n x(n); z) = \mathcal{X}(\frac{z}{a})$, *where* $D = \{|a| \cdot \alpha < |z| < |a| \cdot \beta\}$;
5. $\mathcal{Z}(nx(n); z) = -z\frac{d}{dz}\mathcal{X}(z)$, *where* $D = D_1$;
6. $\mathcal{Z}((x * y)(n); z) = \mathcal{X}(z)\mathcal{Y}(z)$, *where* $D = D_1 \cap D_2$. *Here*

$$(x * y)(n) = \sum_{k=0}^{n} x(k)y(n-k), \quad n \in \mathbb{Z}.$$

Note that in the shift to the left (item 2) it is assumed that $x(-1), \ldots, x(-n_0)$ are non-zero even though the sequence is causal. If these numbers are not specifically given, then they are assumed to be zero.

Example

1. To determine the $\mathcal{Z}$-transform of the causal sequence $x(n) = na^n$, $n \in \mathbb{N}$, the fifth item in the above statement and the tables can be used:

$$\mathcal{Z}(na^n; a) = -z\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Z}(a^n; z) = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{z-a}\right) = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(1 + \frac{a}{z-a}\right) = \frac{az}{(z-a)^2}.$$

2. To determine the $\mathcal{Z}$-transform of the causal sequence $y(n) = a^{n-2}$, note that $y(n)$ is a shift of the causal sequence $x(n) = a^n$ with initial conditions $x(-1) = a^{-1}$ and $x(-2) = a^{-2}$: $y(n) = x(n-2)$. Hence, by item 2 of the above Proposition with $n_0 = 2$,

$$\begin{aligned}\mathcal{Y}(z) &= \mathcal{Z}(y(n); z) = \mathcal{Z}(x(n-2); z) = z^{-2}\mathcal{X}(z) + z^{-2}\textstyle\sum_{m=-2}^{-1} x(m)z^{-m} \\ &= z^{-2}\tfrac{z}{z-a} + z^{-2}\left(a^{-2}z^2 + a^{-1}z\right) = \tfrac{1}{z(z-a)} + a^{-2} + a^{-1}z^{-1}.\end{aligned}$$

Here again we used that $\mathcal{X}(z) = \frac{z}{z-a}$, which can be found from the tables.

Example

Determine the inverse $\mathcal{Z}$-transform of $\mathcal{X}(z) = z^2/((z-a)(z-b))$, where $a \neq b$. Note that with $\mathcal{U}(z) = z/(z-a)$ and $\mathcal{V}(z) = z/(z-b)$ one has that

$$u(n) := \mathcal{Z}^{-1}(\mathcal{U}(z); n) = a^n \quad \text{and} \quad v(n) := \mathcal{Z}^{-1}(\mathcal{V}(z); n) = b^n.$$

Hence, by the convolution statement in Proposition 25

$$\begin{aligned} \mathcal{Z}^{-1}(\mathcal{X}(z); n) &= \mathcal{Z}^{-1}(\mathcal{U}(z) \cdot \mathcal{V}(z); n) = \mathcal{Z}^{-1}(\mathcal{Z}(u(n); z) \cdot \mathcal{Z}(v(n); z); n) \\ &= \mathcal{Z}^{-1}(\mathcal{Z}((u * v)(n); z); n) = (u * v)(n) = \textstyle\sum_{k=0}^{n} u(k)v(n-k) \\ &= \textstyle\sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = b^n \frac{(a/b)^{n+1}-1}{a/b-1} = \frac{a^{n+1}-b^{n+1}}{a-b}. \end{aligned}$$

Another way to solve the problem would be to use partial fractions. Therefore observe that

$$\frac{z^2}{(z-a)(z-b)} = \frac{b}{b-a}\frac{z}{z-b} - \frac{a}{b-a}\frac{z}{z-a}.$$

Hence, using the linearity of the inverse $\mathcal{Z}$-transform

$$\begin{aligned} \mathcal{Z}^{-1}\left(\tfrac{z^2}{(z-a)(z-b)}; n\right) &= \mathcal{Z}^{-1}\left(\tfrac{b}{b-a}\tfrac{z}{z-b} - \tfrac{a}{b-a}\tfrac{z}{z-a}; n\right) = \tfrac{b}{b-a}\mathcal{Z}^{-1}\left(\tfrac{z}{z-b}; n\right) - \tfrac{a}{b-a}\mathcal{Z}^{-1}\left(\tfrac{z}{z-a}; n\right) \\ &= \tfrac{b}{b-a} b^n - \tfrac{a}{b-a} a^n = \tfrac{b^{n+1}-a^{n+1}}{b-a}. \end{aligned}$$

Finally, the inverse $\mathcal{Z}$-transform can also be calculated by means of residue calculus. Therefore note that $\mathcal{X}(z)$ is analytic in $\mathbb{C}$ except for poles (of order one) at a and b. Therefore

$$\mathcal{Z}^{-1}\left(\tfrac{z^2}{(z-a)(z-b)}; n\right) = \operatorname{Res}_b\left(\tfrac{z^2 z^{n-1}}{(z-a)(z-b)}\right) + \operatorname{Res}_a\left(\tfrac{z^2 z^{n-1}}{(z-a)(z-b)}\right) = \tfrac{b^2 b^{n-1}}{b-a} + \tfrac{a^2 a^{n-1}}{a-b} = \tfrac{b^{n+1}-a^{n+1}}{b-a}.$$

The $\mathcal{Z}$-transform: Solving difference equations.

Example

Solve the (initial value) difference equation

$$y(n+2) + 3y(n+1) + 2y(n) = 0, \quad y(0) = 1, y(1) = -3.$$

This problem will be solved by using the $\mathcal{Z}$-transform. Therefore note that

$$\begin{aligned} \mathcal{Z}(y(n+2);z) &= z^2\mathcal{Z}(y(n);z) - z^2 y(0) - z^1 y(1) = z^2\mathcal{Z}(y(n);z) - z^2 + 3z; \\ \mathcal{Z}(y(n+1);z) &= z^1\mathcal{Z}(y(n);z) - z^1 y(0) = z\mathcal{Z}(y(n);z) - z. \end{aligned}$$

Hence, since the $\mathcal{Z}$-transform of 0 is 0, $\mathcal{Z}$-transforming the difference equation yields

$$\begin{aligned} 0 = \; & \mathcal{Z}(y(n+2) + 3y(n+1) + 2y(n);z) \\ = \; & \mathcal{Z}(y(n+2);z) + 3\mathcal{Z}(y(n+1);z) + 2\mathcal{Z}(y(n);z) \\ = \; & z^2\mathcal{Z}(y(n);z) - z^2 + 3z + 3\left(z\mathcal{Z}(y(n);z) - z\right) + 2\mathcal{Z}(y(n);z) \\ = \; & (z^2 + 3z + 2)\mathcal{Z}(y(n);z) - z^2. \end{aligned}$$

Therefore $\mathcal{Z}(y(n);z)$ is given by

$$\mathcal{Z}(y(n);z) = \frac{z^2}{z^2 + 3z + 2} = \frac{z^2}{(z+2)(z+1)}.$$

Consequently, it follows from the calculated results that

$$y(n) \quad = \mathcal{Z}^{-1}\left(\tfrac{z^2}{(z+2)(z+1)};n\right) = \tfrac{(-2)^{n+1} - (-1)^{n+1}}{-2-(-1)} = (-1)^{n+1} - (-2)^{n+1}.$$

Finally, check whether obtained the solution is correct!

Example

Solve the (initial value) difference equation

$$y(n) - y(n-2) = \delta(n-1), \quad y(-1) = 0, y(-2) = 0.$$

This problem will be solved by using the $\mathcal{Z}$-transform. Therefore note that

$$\mathcal{Z}(y(n-2); z) = z^{-2}\mathcal{Z}(y(n); z) + z^{-2}(y(-1)z + y(-2)z^2) = z^{-2}\mathcal{Z}(y(n); z).$$

Hence, $\mathcal{Z}$-transforming the difference equation yields

$$\begin{aligned} \mathcal{Z}(\delta(n-1); z) &= \mathcal{Z}(y(n) - y(n-2); z) = \mathcal{Z}(y(n); z) - \mathcal{Z}(y(n-2); z) \\ &= \mathcal{Z}(y(n); z) - z^{-2}\mathcal{Z}(y(n); z) = \tfrac{z^2-1}{z^2}\mathcal{Z}(y(n); z). \end{aligned}$$

Therefore $\mathcal{Z}(y(n); z)$ is given by

$$\begin{aligned} \mathcal{Z}(y(n); z) &= \tfrac{z^2}{z^2-1}\mathcal{Z}(\delta(n-1); z) = \tfrac{z^2}{(z-1)(z+1)}\mathcal{Z}(\delta(n-1); z) \\ &= \mathcal{Z}\left(\tfrac{1^{n+1}-(-1)^{n+1}}{1-(-1)}; z\right)\mathcal{Z}(\delta(n-1); z) = \mathcal{Z}\left((1-(-1)^{n+1})/2 * \delta(n-1); z\right). \end{aligned} .$$

Consequently,

$$y(n) = \sum_{m=0}^{n} \frac{1-(-1)^{n-m+1}}{2}\delta(m-1) = \begin{cases} 0, & n \le 0; \\ \frac{1-(-1)^n}{2}, & n > 0. \end{cases}$$

Finally, check whether obtained the solution is correct!