Liite A: Fourier transformation

A.1 Fourier series

The 2L-periodic Fourier series of the function $f(t)$

In trigonometric form

$$f(t) \sim S_f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}t) + b_n \sin(\frac{n\pi}{L}t)),$$

missä

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\mathrm{d}t, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi}{L}t)\mathrm{d}t, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi}{L}t)\mathrm{d}t.$$

In exponential or complex form

$$f(t) \sim S_f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}t},$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\frac{n\pi}{L}t}\mathrm{d}t$$

In particular,

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2} \quad \text{ja} \quad c_{-n} = \overline{c_n}.$$

Parseval's identity

$$\frac{1}{2L}\int_{-L}^{L} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = |c_0|^2 + 2\sum_{n=1}^{\infty}|c_n|^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{2}.$$

A.2 Fourier transformation

$$F(w) = \mathcal{F}(f(t); w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\mathrm{d}t$$

Inverse transform	$f(t) = \mathcal{F}^{-1}(F(w); t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{itw}\mathrm{d}w$
Linearity	$\mathcal{F}(af(t) + bg(t); w) = a\mathcal{F}(f(t); w) + b\mathcal{F}(g(t); w)$
Scaling	$\mathcal{F}(f(at); w) = \frac{1}{a}\mathcal{F}(f(t); \frac{w}{a}) \qquad (a > 0)$
Shifting	$\mathcal{F}(f(t - t_0); w) = e^{-iwt_0}\mathcal{F}(f(t); w)$
Frequency shift	$\mathcal{F}(f(t)e^{iw_0 t}; w) = \mathcal{F}(f(t); w - w_0)$
Convolution	$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t - s)\mathrm{d}s = \int_{-\infty}^{\infty} f(t - s)g(s)\mathrm{d}s$
Convolution theorem	$\mathcal{F}((f * g)(t); w) = \mathcal{F}(f(t); w)\mathcal{F}(g(t); w)$
Transform of derivative	$\mathcal{F}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t); w\right) = (iw)^n\mathcal{F}(f(t); w)$
Transform of primitive	$\mathcal{F}\left(\int_{-\infty}^{t} f(s)\,\mathrm{d}s; w\right) = \dfrac{\mathcal{F}(f(t); w)}{iw} + \pi\delta(w)\mathcal{F}(f(t); 0)$
Parseval's identity	$\int_{-\infty}^{\infty} \lvert f(t)\rvert^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert\mathcal{F}(f(t); w)\rvert^2\mathrm{d}w = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(w)\rvert^2\mathrm{d}w$

A.3 Discrete Fourier transform (DFT)

$$X(k) = \mathfrak{F}(x(n); k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-ink\frac{2\pi}{N}}$$

Inverse transformation	$x(n) = \mathfrak{F}^{-1}(X(k); n) = \sum_{k=0}^{N-1} X(k)e^{ink\frac{2\pi}{N}}$
Linearity	$\mathfrak{F}(ax(n) + by(n); k) = a\mathfrak{F}(f(n); k) + b\mathfrak{F}(y(n); k)$
Shifting	$\mathfrak{F}(x(n - n_0); k) = e^{-in_0 k\frac{2\pi}{N}}\mathfrak{F}(x(n); k)$
Modulation	$\mathfrak{F}\left(e^{ink_0\frac{2\pi}{N}} x(n); k\right) = \mathfrak{F}(x(n); k - k_0)$
Convolution	$(x * y)(n) = \sum_{k=0}^{N-1} x(k)y(n - k)$
Convolution theorem	$\mathfrak{F}(x(n); k)\mathfrak{F}(y(n); k) = \mathfrak{F}(\frac{1}{N}(x * y)(n); k)$

Function	Fourier transform
$\begin{cases} 1, & b<t<c \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{-ibw}-e^{-icw}}{iw}$
$\begin{cases} 1, & -c<t<c \\ 0, & \text{otherwise} \end{cases}$	$\frac{2\sin(cw)}{w}$
$e^{-a\|t\|}, \quad a>0$	$\frac{2a}{w^2+a^2}$
$e^{-at^2}, \quad a>0$	$\sqrt{\frac{\pi}{a}}\, e^{-\frac{w^2}{4a}}$
$\frac{\sin(at)}{t}, \quad a>0$	$\begin{cases} \pi, & \|w\|<a \\ 0, & \|w\|>a \end{cases}$
$\frac{1}{t^2+a^2}, \quad a>0$	$\frac{\pi}{a}\, e^{-a\|w\|}$
$\delta(t)$	1
1	$2\pi\delta(w)$
e^{iw_0t}	$2\pi\delta(w-w_0)$
$s(t)=\begin{cases} -1, & t<0 \\ 1, & t>0 \end{cases}$	$\dfrac{2}{iw}$
$u(t)=\begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$	$\dfrac{1}{iw}+\pi\delta(w)$
$u(t)e^{-at}, \quad a>0$	$\frac{1}{a+iw}$
$\cos(at)$	$\pi\left[\delta(w-a)+\delta(w+a)\right]$
$\sin(at)$	$-i\pi\left[\delta(w-a)-\delta(w+a)\right]$

Taulukko A.1: Fourier transforms.

Liite B: Laplace transformation

$$F(s) = \mathcal{L}(f(t); s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

Inverse transform	$f(t) = \mathcal{L}^{-1}(F(s); t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{ts}\mathrm{d}s$ $= \frac{1}{2\pi i}\oint_C F(s)e^{ts}\mathrm{d}s = \sum_{a_j \text{ pole of } F} \mathrm{Res}_{s=a_j}\left\{F(s)e^{st}\right\}$
Linearity	$\mathcal{L}(af(t) + bg(t); s) = a\mathcal{L}(f(t); w) + b\mathcal{L}(g(t); s)$
Scaling	$\mathcal{L}(f(at); s) = \frac{1}{a}\mathcal{L}(f(t); \frac{s}{a}) \quad (a > 0)$
Shifting (to the right)	$\mathcal{L}(u(t-t_0)f(t-t_0); s) = e^{-st_0}\mathcal{L}(f(t); s)$, where $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$ ja $t_0 \geq 0$
Multiplying with exponential function	$\mathcal{L}(e^{at}f(t); s) = \mathcal{L}(f(t); s-a)$
Transform of the derivative	$\mathcal{L}(\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t); s) = s^n\mathcal{L}(f(t); s) - s^{n-1}f(0)$ $-s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
Transform of the primitive	$\mathcal{L}\left(\int_0^t f(u)\mathrm{d}u; s\right) = \frac{1}{s}\mathcal{L}(f(t); s)$
Multiplying with potentials	$\mathcal{L}((t^n f(t); s) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}(f(t); s), \quad n = 0, 1, 2, \ldots$
Primitive of the Laplace transform	$\mathcal{L}(\frac{1}{t}f(t); s) = \int_s^\infty F(u)\mathrm{d}u$
Transform of periodic function	$\mathcal{L}(f(t); s) = \frac{\int_0^T f(t)e^{-st}\mathrm{d}t}{1-e^{-sT}}, \quad f(t+T) = f(t)$
Convolution	$(f * g)(t) = \int_0^t f(s)g(t-s)\mathrm{d}s$
Convolution theorem	$\mathcal{L}((f * g)(t); s) = \mathcal{L}(f(t); s)\mathcal{L}(g(t); s)$

In Table B.1 the Laplace transform is defined for the values $\mathrm{Re}\, s > 0$, unless otherwise stated.

Function	Laplace transform
$\delta(t)$	1
1 tai $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}, \quad (\mathrm{Re}\, s > a)$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$1 - \cos(at)$	$\frac{a^2}{s(s^2+a^2)}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}, \quad (\mathrm{Re}\, s > \max\{0, a\})$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad (\mathrm{Re}\, s > a)$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad (\mathrm{Re}\, s > a)$

Taulukko B.1: Laplace transforms.

Liite C: $\mathcal{Z}$ Transformations

The two-sided $\mathcal{Z}$ transform is given by

$$X(z) = \mathcal{Z}(x(n); z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

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$$X(z) = \mathcal{Z}(x(n); z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Inverse transform	$x(n) = \mathcal{Z}^{-1}(X(z); n) = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} \mathrm{d}z$
Linearity	$\mathcal{Z}(ax(n) + by(n); z) = a\mathcal{Z}(x(n); z) + b\mathcal{Z}(y(n); z)$ $D = D_1 \cap D_2$
One-sided shift (right)	$\mathcal{Z}(x(n - n_0); z) = z^{-n_0} \mathcal{Z}(x(n); z) + z^{-n_0} \sum_{k=-n_0}^{-1} x(k) z^{-k}$ $D = D_1$ ja $n_0 > 0$
One-sided shift (left)	$\mathcal{Z}(x(n + n_0); z) = z^{n_0} \mathcal{Z}(x(n); z) - z^{n_0} \sum_{k=0}^{n_0-1} x(k) z^{-k}$ $D = D_1$ ja $n_0 > 0$
Two-sided shift	$\mathcal{Z}(x(n - n_0); z) = z^{-n_0} \mathcal{Z}(x(n); z), \quad D = D_1$
Multiplying with potentials	$\mathcal{Z}(a^n x(n); z) = \mathcal{Z}\left(x(n); \frac{z}{a}\right), \quad D = \{\lvert a \rvert \alpha < \lvert z \rvert < \lvert a \rvert \beta\}$
Multiplying with n	$\mathcal{Z}(nx(n); z) = -z \frac{\mathrm{d}}{\mathrm{d}z} \mathcal{Z}(x(n); z), \quad D = D_1$
Convolution	$(x * y)(n) = \sum_{k=0}^{n} x(k) y(n-k)$
Convolution theorem	$\mathcal{Z}((x * y)(n); z) = \mathcal{Z}(x(n); z) \mathcal{Z}(y(n); z), \quad D = D_1 \cap D_2$

Series	$\mathcal{Z}$ transform	Region of convergence
$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\frac{z}{z-1}$	$D = \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$a^n u(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\frac{z}{z-a}$	$D = \{z \in \mathbb{C} : \lvert z\rvert > \lvert a\rvert\}$
$nu(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\frac{z}{(z-1)^2}$	$D = \{z \in \mathbb{C} : \lvert z\rvert > 1\}$
$na^n u(n) = \begin{cases} na^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$\frac{az}{(z-a)^2}$	$D = \{z \in \mathbb{C} : \lvert z\rvert > \lvert a\rvert\}$
$\begin{cases} \frac{1}{n!}, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$e^{\frac{1}{z}}$	$D = \mathbb{C} \setminus \{0\}$
$\delta(n)$	1	$D = \mathbb{C}$
$\delta(n - n_0)$	z^{-n_0}	$D = \mathbb{C}$, $n_0 \leq 0$, ja $D = \mathbb{C} \setminus \{0\}$, $n_0 > 0$

Taulukko C.1: $\mathcal{Z}$ transforms.

Liite D: Complex analysis

Some formulas from complex analysis:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad n = 0, 1, 2, \ldots$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right) \right].$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{z \in \mathbb{C}_+ \text{ pole}} \mathrm{Res}\ f(z)$$