Use of Modern Portfolio Theory in Systematic Portfolio Strategies

Seppo Pynnönen,[1] Ph.D.

Department of Mathematics and Statistics,

University of Vaasa, P.O.Box 700, Vaasa Finland, e-mail sjp@uwasa.fi

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Abstract

This paper considers active portfolio management under the assumptions that short selling is not allowed, or is otherwise inconvenient to carry out. Under these assumptions the fractions of market return and risk are incorporated as parts of the total excess return and risk that are due to the active management. The necessary formulas are derived under the assumption that the quadratic utility function serves a reasonable approximation to the investor's preferences. Both portfolio and share level evaluation tools are given. An investor can exploit these tools in evaluation whether the current portfolio does optimally match his/her willingness to bear risk as a cost for extra return from active portfolio management.

1. Introduction

Modern portfolio theory (MPT) serves the scientific approach to investment. The basic theory is established by the pioneering work of Harry Markowitz (1959). Latest developments on this area can be found from Rudd and Clasing (1982), Elton and Gruber (1991), Markowitz (1991), and Markowitz, Schaible and Ziemba (1992) [a short review in Finnish about principles is given Pynnönen (1990)]. The key idea of the theory is the attempt to model the relation between risk and reward. With the model an investor can quantitatively maximize the expected reward consistent with his or her willingness to bear risk.

Treynor and Black (1972), Rosenberg (1979), Rudd and Clasing (1982), and Pynnönen (1990) consider active portfolio strategies suitable for markets where short selling is allowed. This makes it possible to consider residual return and market return of a portfolio as two independent portfolios. Accordingly managing of them can be considered as independent business. Nevertheless, in some markets, direct short selling is prohibited by law or is otherwise inconvenient to carry out. Consequently the independence of the residual and market returns does not necessarily hold any more. This implies that the excess return and risk gained by the active management contains besides the residual components also a fraction of market risk and market return, due to the change in the market position as a consequence of reweighting the original portfolio.

The aim of the present paper is to contribute the current literature of active portfolio management by incorporating the fraction of market return and risk as parts of the total excess return and risk due to the active management. This theory applies markets where short selling is not allowed by law or it is otherwise inconvenient to carry out in portfolio management. The necessary formulas are derived in Chapters 4 and 5. Otherwise, the content of the paper is as follows: In Chapter 2 the basic model for security returns is fixed. In Chapter 3 the key concepts in portfolio management logic are defined. In Chapter 4 portfolio monitoring formulas are derived, and in Chapter 5 tools for evaluating market timing and sharewise analysis as well as stock picking are given. Chapter 6 concludes.

2. Risk and reward

A crucial observation in modern portfolio theory is that security prices tend to move more or less in unison. Furthermore, it has been observed that within a branch of industry securities tend to move more in unison than between branches. This has led theoreticians to the natural conclusion that there exist common *factors* in the markets that are causes of the observed co-movement of securities. For instance, there may be factors associated with homogeneous industries, growth stocks or small companies. Nevertheless, not all of the variation in security prices can be predicted by the common factor, but some of the variation is unique to individual securities.

Calling the influence of common factors as the *systematic* part and the rest as the *residual* part, we obtain the following decomposition for the rate of return of a security

$$R = g(f_1, \ldots, f_k, \epsilon) \tag{2.1}$$

where R is the rate of return of a security, f_i is the ith common factor [$i = 1, \dots, k$ (= number of common factors)], ϵ is the residual term, unique to the security in question, and g is a real valued function giving the rule how the common factors and the residual term are related to determine the return R. In practice, however, the exact form of g is unknown, and a linear approximation is used.

Here we shall adopt the simplest case by considering only one common factor, the *market factor*, whose return is denoted by R_m. For a detailed description and properties of the single and multiple factor model the interested reader is referred to e.g. Elton and Gruber (1991).

Before continuing, we define more precisely some technical details. The rate of return for a security in time period t is defined as

$$R_t = \log(P_t + \mathrm{div}_t) - \log(P_{t-1}), \tag{2.2}$$

where P_t is security's price at the end of period t, and div_t denotes the cash dividend. Furthermore, let r_f denote the risk free interest rate, then the difference

$$r = R - r_f \tag{2.3}$$

is called the *excess* rate of return of the associated security.

Given these conventions the single factor or single index model becomes

$$r = \alpha + \beta(R_m - r_f) + \epsilon, \tag{2.4}$$

where $R_m - r_f$ denotes the market excess rate of return, and α and β are constants. In (2.4) r, R_m and ϵ are random variables with $E(\epsilon) = 0$, where E stands for the expectation operator, and ϵ and R_m are uncorrelated. Parameter α indicates the mean residual rate of return, and β the coefficient how the expected value of r is related to the markets.

It may be noted that, if the true model is a multiple factor model with k factors of the form

$$r = \beta_1 F_1 + \cdots + \beta_k F_k + \epsilon', \tag{2.5}$$

where ϵ' is independent of the factors F_j and $E(\epsilon') = 0$. Then defining $F_1 = R_m - r_f$, $\alpha = E(\beta_2 F_2 + \cdots \beta_k F_k)$ and $\epsilon = \epsilon' + [\beta_2(F_2 - E(F_2)) + \cdots \beta_k(F_k - E(F_k))]$, we obtain the market model given by (2.4). Hence, observing nonzero alphas and nonzero correlations between epsilon terms of different stocks can be interpreted as influence of some temporal *extra market* effect on these stocks. In fact, the main task of portfolio managers is to identify the effects of these factors before they actually occur in order to earn extra profit. We shall return to these questions in the course of this study.

Using the variance as a risk measure, we obtain from (2.4)

$$\sigma^2 = \beta^2 \sigma_m^2 + \omega^2, \tag{2.6}$$

where σ^2, σ_m^2 and ω^2 are the variances of r, R_m and ϵ, respectively. Hence, the total risk of the rate of return of a security decomposes to the market risk, $\beta^2\sigma_m^2$, and residual risk ω^2 that is unique to the individual security.

Compiling n securities, r_i $(i = 1, \ldots, q)$, into a portfolio P, where the ith security has the weight h_i with $h_i \geq 0, h_1 + \cdots + h_q = 1$, we obtain by (2.4) the following model for the excess rate of return of the portfolio,

$$r_p = \alpha_p + \beta_p(R_m - r_f) + \epsilon_p, \tag{2.7}$$

where $r_p = h_1 r_1 + \cdots + h_q r_q$, $\alpha_p = h_1\alpha_1 + \cdots + h_q\alpha_q$, $\beta_p = h_1\beta_1 + \cdots + h_q\beta_q$, and $\epsilon_p = h_1\epsilon_1 + \cdots + h_q\epsilon_q$.

Expected return and variance get the forms

$$\mu_p = \alpha_p + \beta_p(\mu_m - r_f) \tag{2.8}$$

$$\sigma_p^2 = \beta_p^2\sigma_m^2 + \omega_p^2, \tag{2.9}$$

where μ_p and μ_m are the expected values of r_p and r_m respectively, and σ_p^2, σ_m^2 and ω_p^2 are the variances of r_p, R_m and ϵ_p, respectively.

Defining $h_i^a = h_i - \beta_p w_i$ as the *adjusted active holding* of the ith share in an portfolio, the residual variance, ω_p^2, and the abnormal return, α_p can be written as (see Appendix for further details)

$$\omega_p^2 = \sum_{i=1}^{Q}(h_i^a)^2\omega_i^2 + 2\sum_{i=1}^{Q}\sum_{j<i} h_i^a h_j^a \omega_{ij}, \tag{2.10}$$

$$\alpha_p = \sum_{i=1}^{Q} h_i^a \alpha_i \tag{2.11}$$

where ω_{ij} is the covariance of ϵ_i and ϵ_j, $\omega_i^2 = \mathrm{var}(\epsilon_i)$ is the variance of ϵ_i, $i, j = 1, \ldots, Q$ (= number of all shares in the market). Note that in the market portfolio $h_i^a = 0$ for all i, and hence the logically necessary properties for a market portfolio are fulfilled, i.e., $\omega_m^2 = 0$, and $\alpha_m = 0$.

Decompositions (2.7)–(2.11) are the building blocks for the three step strategy sketched in the subsequent chapters.

3. Normal active and systematic portfolios

The flow of information is the basic power steering the behavior of investors. Hence, to construct a systematic approach to respond that information, it is reasonable to structure the information flows. The starting point is the situation that is stable in the sense that nothing special is occurring in the markets. The corresponding portfolio that optimally responds to this market state gives a natural benchmark against what one can compare his/her current portfolio.

This basic starting portfolio is usually called a *normal* portfolio. The construction of this portfolio will be considered more thoroughly in the next chapter. For the moment it is enough to know that it is the basic combination of risky shares, consistent with the investor's preferences.

After constructing the normal portfolio one has to develop a managing strategy that defines the rules how to bias the normal portfolio in response to the information flow that probably will change the return of some securities. The difference between the revised and normal portfolio is called an *active* portfolio. Thus if the holding of the ith security in the normal portfolio is h_i, and in the current portfolio, h_i^c, then the difference, $h_i^a = h_i^c - h_i$, is called the *active holding*. This definition slightly differs from the technical adjusted active holding concept defined earlier. The main question then is, how large difference should be carried out in response to the information. These points are considered in Chapters 4 and 5.

Finally, we note that for technical purposes one may define the so called *systematic* portfolio [cf., e.g. Pynnönen (1990), Rudd and Clasing (1982)]. With this it is usually meant a benchmark index against what an investor can compare and monitor the success and ability of his/her portfolio manager. Usually some general index like HEX in Finland serves such a benchmark. Also some other mixes of shares are possible. For example investor can personally define a suitable mix of shares which he uses as a benchmark. This approach is especially useful, if there are legal restrictions in the amount or type (e.g., restricted shares) shares that can be owned. In these situations it is natural to restrict the systematic portfolio into the same subset of shares. Nevertheless, for the sake of simplicity we shall not make conceptual distinction between the market portfolio and the systematic portfolio, but use only the term market portfolio.

4. Selection of normal and active portfolio

Selection of normal and active portfolios depends on investor's personal preferences. A basic assumption in MPT is that investors prefer more to less and are risk averters. In short, investor's utility function is upwards sloping and concave with respect to wealth. The combination of risk and return that keeps the utility unchanged defines an indifference curve. Change in utility occurs only by moving to another indifference curve.

In this construction we have implicitly assumed, following the practice of the usual MPT framework, that two statistics, mean (μ) and variance (σ^2), are adequate to characterize return distributions. Using the standard deviation as the risk measure we make one further simplification and assume that the indifference curves in (σ, μ) space are given by the curves

$$C(\mu, \sigma) = \mu - \lambda\sigma^2, \tag{4.1}$$

where λ (> 0) is a parameter unique to each investor, called the *risk aversion parameter*. It may be noted that the selection of the form of an utility function is not a crucial matter (see e.g. Chopra and Ziemba (1993), p. 6), hence although the quadratic utility function has well known drawbacks, it can be expected to be a useful approximation for the true utility function.

4.1 Normal portfolio

Under the assumptions of the capital asset pricing model (CAPM) the market portfolio is efficient in the sense that its expected return is at least as much as any other portfolio in the same risk class. Furthermore, all unsystematic risk is diversified out and there is no possibility of extra market gain in the market portfolio. That is in the notations of (2.7)–(2.9), $\alpha_m = 0$, $\epsilon_m = 0$, $\omega_m = 0$, and furthermore $\beta_m = 1$. However, as we mentioned earlier this portfolio is not attainable, for it contains all possible securities people can imagine to acquire. Hence, for practical purposes a proxy must be used. In our case a suitable proxy is the Helsinki Stock Exchange general index, HEX.

Adopting this convention, the selection of investor's normal portfolio reduces to the question of what fraction of the available funds the investor is going to invest into the markets (the rest is invested in the risk free alternative). This is a question of preferences that define how much the investor is willing to bear market risk, σ_m (which is the only risk as was seen in the previous paragraph).

Let the fraction invested into the markets be denoted by β_n. Next we shall see how this is related to an investor's risk aversion parameter, λ, when the market risk, σ_m, and the market expected excess return, μ_m, are given.

From (2.7) we observe that the excess return, r_n, of the normal portfolio becomes $r_n = \beta_n r_m$, and hence, the expected value is $\mu_n \equiv E(r_n) = \beta_n \mu_m$. The variance of r_n becomes $\sigma_n^2 = \beta_n^2 \sigma_m^2$, from which we get $\beta_n = \sigma_n / \sigma_m$.

Now considering σ_m and μ_m as constants then we can write

$$\mu = \frac{\mu_m}{\sigma_m}\sigma. \tag{4.2}$$

The utility of an investor is the higher the higher indifference curve he/she can reach in the (σ, μ) plane. Rewriting (4.1) we get $\mu = C + \lambda\sigma^2$. Hence, we can write the indifference curve only as a function of the risk parameter, σ, as $C = \sigma\mu_m/\sigma_m - \lambda\sigma^2$. The maximum is found from the zero point of the derivative of the utility function with respect to σ, i.e., when $dC/d\sigma = \mu_n/\sigma_m - 2\lambda\sigma = 0$. Solving for the riskaversion parameter, λ, we get $\lambda = \mu/2\sigma^2$. Noting further that $\mu = \beta\mu_m$, and $\sigma = \beta\sigma_m$, we get finally at $\sigma = \sigma_n$, and $\mu = \beta_n\mu_m$ (with $\beta_n = \sigma_n/\sigma$)

$$\lambda = \frac{\mu_m}{2\beta_n\sigma_m^2}. \tag{4.3}$$

Hence, if one knows how the investor has allocated his money between the risky market and the risk free alternative, then one can instantaneously infer the investor's preference structure. Conversely, if the risk aversion parameter is known then the normal portfolio can be constructed.

As we shall see in the next chapter, formula (4.3) gives a powerful result; that knowing β_n determines also the optimal active strategy that should be performed in response to the future prospect in the market conditions.

Finally, let us define the *mean-variance ratio* or the *return too risk* ratio for a portfolio, P, as

$$\theta_p = \frac{\mu_p}{\sigma_p^2}. \tag{4.4}$$

Then we observe from (4.3) that $\theta_n = \mu_n/\sigma_n^2 = \mu_m/\beta_n\sigma_m^2 = 2\lambda$. Because the risk aversion parameter for an investor is a constant, we have a fundamental result: *In the optimum the portfolio must have a mean-variance ratio equal to twice the investor's risk aversion parameter*, i.e., the following relation must hold

$$\frac{\mu}{\sigma^2} = 2\lambda. \tag{4.5}$$

From this simple relation we can derive all the key characteristics of a portfolio.

4.2 Active portfolio

In Chapter 3 the active portfolio was defined as the difference of the current and the normal portfolio. In this chapter we shall derive the parameters to control the amount of aggressiveness in reweighting the current portfolio in response to new information so that the outcome best matches the investor's preferences.

The key components in active management are the security alphas, α_i, Betas, β_i, residual variances, ω_i^2, and portfolio's Beta, β_p.

Suppose the portfolio manager responds to a new piece of information by revising the normal portfolio, N, to a portfolio P. Then the rate of return of the revised portfolio is of the form (2.7). Analysis of the consequences of the revision can be started from this model [see Rosenberg (1975), Rudd and Rosenberg (1980), Rudd and Clasing (1982), Pynnönen (1990)].

If short selling is allowed then the systematic return and the residual return can be considered as returns of two independent portfolios, because the market position accumulated as a by-product in taking positions in individual securities can be eliminated by taking a suitable opposite position in the market security. Consequently the optimality of the active strategy can be purely judged on the basis of the residual risk, ω_p, and the abnormal return, α_p. Nevertheless, when short selling is not possible then these two portfolios cannot be always made independent. Consequently changes in the portfolio induce usually also a change in the market position, which must be taken account of in the active strategy. We shall sketch the necessary steps to do this adapting the derivations of Treynor and Black (1973), and Rosenberg (1979).

Let $A = P - N$ denote th active portfolio, then the excess return due to the policy can be written as

$$r_A = r_p - r_n, \tag{4.6}$$

where r_A denotes the excess return due to the active management. Using (2.7) and the relation $r_n = \beta_n r_m$, we can write

$$r_A = \alpha_p + (\beta_p - \beta_n) r_m + \epsilon_p, \tag{4.7}$$

where β_p is the Beta of the revised portfolio.

It would be tempting to analyze the expected return and risk of (4.6) of its own, but this is not the correct way, because the residual portfolio and the new market position of the revised portfolio in our framework are not necessarily independent. Hence, we must solve the problem via considering the total return and total risk. The expected total return is

$$\mu_p = \alpha_p + \beta_p \mu_m \tag{4.8}$$

and the total variance is

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \omega_p^2. \tag{4.9}$$

In order to meet optimality with respect to the investor's preferences the new portfolio should satisfy the mean variance relation given by (4.5)

$$2\lambda = \frac{\mu_p}{\sigma_p^2} = \frac{\beta_p \mu_m + \alpha_p}{\sigma_p^2}. \tag{4.10}$$

But $2\lambda = \mu_m / \beta_n \sigma_m^2$. Equating this with (4.10), and solving for α_p gives the required rate that *must* be gained by the new policy. Denoting this required alpha as α_p^R we obtain

$$\begin{aligned} \alpha_p^R = \frac{\sigma_p^2}{\beta_n \sigma_m^2} \mu_m - \beta_p \mu_m &= \frac{\omega_p^2}{\beta_n \sigma_m^2} \mu_m + (\frac{\beta_p}{\beta_n} - 1) \beta_p \mu_m \\ &= \alpha_p^{\mathrm{RS}} + \alpha_p^{\mathrm{RM}}, \end{aligned} \tag{4.11}$$

where $\alpha_p^{\mathrm{RS}} = \mu_m \omega_p^2 / \beta_n \sigma_m^2$ stands for the required alpha due to the increased residual risk, and $\alpha_p^{\mathrm{RM}} = (\beta_p/\beta_n - 1)\beta_p \mu_m$ stands for the required excess return due to the changed market position. Note that if β_p is close to β_n the last term on the right hand side of (4.11) disappears, which is the approximation given in Pynnönen (1990).

5. Active strategy

Basically active strategy means responding to market information by a mix of *market timing* and *stock reweighting*. In order to be optimal the strategy must follow the preferences of the investor in question. Given the preference structure, i.e., the normal beta, β_n, achieving the optimality can be monitored by the means described in the previous chapter. Let us consider first the theory of market timing.

5.1 Market timing

Here we may assume that the investor allocates his or her money between the markets and the risk free target (e.g. banking account). Then essentially the situation is that the investor has two securities, the market portfolio and the risk free alternative, between witch the funds are swapped in response to the information from expected market conditions.

Suppose that the investor estimates that the excess return will deviate in the next period by an amount e from the expected value μ_m. The question then is: How much should the investor change the current position to respond optimally to the information that is assumed reliable? To answer this rewrite (4.3) as

$$\beta_n = \frac{\mu_m}{2\lambda\sigma_m^2}. \tag{5.1}$$

Taking the estimated excess return for the next period into account, we obtain $\mu_m + e$. Hence, to maximize investor's utility, swapping between the risk free target and markets must be such that the structure with the changed beta is of the form (5.1), for it gives the tangent point of investor's indifference curve. The new beta satisfying (5.1) is called the *target beta*, denoted by β_T. Hence, substituting $\mu_m + e$ for μ_m in (5.1) gives

$$\beta_T = \frac{\mu_m + e}{2\lambda\sigma_m^2} = \beta_n + \frac{e}{2\lambda\sigma_m^2}. \tag{5.2}$$

Remembering that $2\lambda = \mu_m/\beta_n\sigma_m^2$, we get finally the optimum policy with respect to the market forecast, e, as

$$\beta_T = \beta_n + \frac{\beta_n e}{\mu_m}. \tag{5.3}$$

Note that here β_T is considered as a stochastic variable for it is dependent on the forecast, e, which is a random variable. In the long run the forecast can be considered independent of the market return, because one cannot expect somebody continuously forecasting the extra market return correctly. Hence the total risk, σ_T^2, from the policy becomes

$$\begin{aligned} \sigma_T^2 &= \mathrm{var}(\beta_T r_m) = \mathrm{var}[(\beta_n + \Delta\beta) r_m] \\ &= \beta_n^2\sigma_m^2 + \mathrm{var}(\Delta\beta)\sigma_m^2, \end{aligned} \tag{5.4}$$

where $\Delta\beta = \beta_T - \beta_n$. Thus the increase in risk is on average equal to $\sigma_m^2 \mathrm{var}(\Delta\beta)$, which, by using the relation $\Delta\beta = \beta_n e/\mu_m$, is more convenient to write for estimation purposes into the form

$$\text{(5.5)} \qquad \text{Risk increase } = \mathrm{var}(\Delta\beta)\sigma_m^2 = \frac{\beta_n^2\sigma_m^2}{\mu_m}\mathrm{var}(e).$$

Term $\mathrm{var}(e) = \mathrm{E}(e^2)$ ($\mathrm{E}(e) = 0$) in (5.5) indicates the error variance in prediction and may be estimated, for example, from analyst's earlier success in predictions. Of course there exist various other (and more sophisticated) possibilities as well.

Now that we have solved the optimal reaction level, β_T, and the imposed risk, with respect to the prediction, we must finally determine the risk adjusted *required return* that should be gained in order to accept the change in the current position. This is obtained straightforwardly using (5.2). The required gain from the policy becomes $\mathrm{E}(r_T - r_n) = \mathrm{E}[\Delta\beta(\mu_m + e)]$, which we shall call the *required target alpha*, and denote by α_T^R. Formally this becomes then

$$\text{(5.6)} \qquad \begin{aligned} \alpha_T^R &= \mathrm{E}[\Delta\beta(\mu_m + e)] = \mathrm{E}(\Delta\beta e) \\ &= \frac{\beta_n}{\mu_m}\mathrm{var}(e), \end{aligned}$$

where we have used (5.3) and that $\mathrm{E}(\Delta\beta) = 0$.

In this simple form it is assumed that the prediction risk $\mathrm{var}(e)$ is independent of the size of the prediction. In practice, however, it may be more convenient to model the risk as an increasing function of the size of the forecast. There are two obvious sources in the prediction uncertainty; one is due to the analyst and the other is pure random error. The analyst's error might be possibly modeled as an increasing function of the prediction.

Finally, it may be noted that market timing is very sensitive to forecasts, e, i.e., small changes in expected market conditions require strong response in order to optimally follow investor's preferences. Furthermore, to fully respond the market changes may require sometimes also short selling. Hence, it cannot be always fully employed in markets where short selling is not possible. The policy also needs use of financial leverage.

5.2 Stock selection

In stock selection the interest is in the future prospects of individual stocks. The main difference in this strategy compared to the previous one is diversification. That is, one can appraise at the same time several stocks and balance the portfolio with them in a certain time point and across time, whereas in market timing the diversification occurs only between different time points. Another main difference is that also the specific risk, (ω^2), comes in.

The most crucial objective in active management is to maintain asset holding at a level where increase in risk can be exactly justified by increase in return. Here we shall consider methods to achieve this goal. However, for the sake of simplicity we shall not take into account taxation and brokers' commissions, although especially in small markets they may be remarkable and bigger than in well developed markets.

For a systematic and logically defendable active strategy the start off point is analysts' forecasts for the current value and risk of each stock under consideration. In the framework of the market model (2.4) this means besides estimation of stock Betas, estimation of alphas, residual variances for each stock, and residual covariances for each pair of stocks.

The next step is to find the optimal balancing in the current portfolio. This can be achieved by portfolio optimization programs. The optimal active portfolio is on the line connecting the origin and a tangent point of the efficient frontier.

Imposing investor's indifference curves into this space the optimal point is found where the frontier is a tangent for an indifference curve.

The next question is whether the suggested active strategy will meet investor's required alpha, α_R^P, given by (4.11). If the expected alpha of the revised portfolio would be less than the required alpha, the suggested change should be reject because the additional risk induced by the new position would not be compensated. If the expected alpha is greater than or equal to the required alpha the revision should be adopted in the current portfolio.

While (4.11) offers an investor means to evaluate whether the new strategy is profitable, it is an aggregate measurement. In order to evaluate stockwise the optimality of the current portfolio we can start off with maximizing (4.1) with respect to holdings h_i. We rewrite (4.1) in terms of (4.8) and (4.9) as

$$C_p = \mu_p - \lambda\sigma_p^2. \tag{5.7}$$

The maximum of C_p is found by setting the partial derivatives, $\partial C_p/\partial h_i$, equal to zero, i.e., $\partial\mu_p/\partial h_i - \lambda\sigma_p^2/\partial h_i = 0$. From (4.8) and (4.9)

$$\frac{\partial\mu_p}{\partial h_i} = \alpha_i + \beta_i\mu_m \tag{5.8}$$

$$\begin{aligned}\frac{\partial\sigma_p^2}{\partial h_i} &= 2\beta_i\beta_p\sigma_m^2 + 2h_i\omega_i^2 + 2\sum_{j\neq i} h_j\omega_{ij} \\ &= 2\omega_p\frac{\partial\omega_p}{\partial h_i} + 2\beta_i\beta_p\sigma_m^2,\end{aligned} \tag{5.9}$$

where ω_p denotes the residual standard deviation of the current portfolio. Equating the right hand side of (5.8) with the right hand side of (5.9) multiplied by λ, and solving for α_i (denoting the solution as α_i^R) gives after taking into account that $2\lambda = \mu_m/\beta_n\sigma_m^2$,

$$\alpha_i^R = \frac{\mu_m}{\beta_n\sigma_m^2}\omega_p\frac{\partial\omega_p}{\partial h_i} + \left(\frac{\beta_p}{\beta_n} - 1\right)\beta_i\mu_m, \tag{5.10}$$

which reminds the aggregate, portfolio level measure (4.11). Actually (4.11) is obtained from (5.10) as a weighted sum. The measures in (5.10) offer a simple, yet powerful, tools to analyze the consistency of portfolio management. A rational investor requires that the benefit from active management covers the costs that are due to management costs, transaction costs, and additional risk of loosing the expected reward.

For the portfolio manager these measures offer means for fine tuning the final portfolio to respond optimally to current alpha estimates (adjusted for risk). Furthermore, given alpha estimates, we can derive the optimal weights for each share by equating the right hand side of (5.10) with the estimates and solving for the portfolio weights, h_i. Denoting these weights by h_i^*, and noting that

$$\frac{\partial\omega_p^2}{\partial h_i} = 2h_i\omega_i^2 + 2\sum_{j\neq i} h_j\omega_{ij}, \tag{5.11}$$

the solutions for h_i^* become

$$h_i^* = \frac{\sigma_m^2}{\omega_i^2}\frac{\beta_n}{\mu_m}\left[\alpha_i - \left(\frac{\beta_p}{\beta_n} - 1\right)\beta_i\mu_m\right] - \sum_{j\neq i} h_j \frac{\omega_{ij}}{\omega_i^2}. \tag{5.12}$$

These solutions require iterative procedure.

6. Conclusions

In this paper we have analyzed active portfolio strategy on a market where short selling is not allowed. Active management has been divided to market timing and stock selection. Starting from the market model, and assuming that the quadratic utility function serves a reasonable approximation for the true utility function for the investors, necessary formulas for active portfolio management were derived.

Due to the short selling restrictions it appears that in the analysis the change in market position must be taken into account, too, when one evaluates the portfolio optimality with respect to investors preferences. Accordingly the required extra return in response to the active management can be split into two components, which show how much of the total required return should come each of the sources.

Finally also the optimal holdings with respect to the alpha estimates were derived.

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Appendix

Let the (excess) return of a portfolio, P, with q securities, obey the market model

$$r_p = \sum_{i=1}^{q} h_i r_i = \sum_{i=1}^{q} h_i(\alpha_i + \beta_i r_i + \epsilon_i) = \alpha_p + \beta_p r_m + \epsilon_p, \tag{A.1}$$

where h_i is the proportion of share i in portfolio, $\alpha_p = \sum_{i=1}^{q} h_i \alpha_i$, $\beta_p = \sum_{i=1}^{q} h_i \beta_p$, r_m is the market excess return, and $\epsilon_p = \sum_{i=1}^{q} h_i \epsilon_i$. Then we have the following results

Proposition A.1 *The abnormal return and residual risk of a portfolio are*

$$\alpha_p = \sum_{i=1}^{Q} h_i^a \alpha_i = \mathrm{E}[\sum_{i=1}^{Q} h_i^a r_i] \tag{A.2}$$

$$\begin{aligned} \omega_p^2 &\equiv \mathrm{var}(\epsilon_p) = \sum_{i=1}^{Q} (h_i^a)^2 \sigma_i^2 + 2\sum_{i<j} h_i^a h_j^a \sigma_{ij} \\ &= \sum_{i=1}^{Q} (h_i^a)^2 \omega_i^2 + 2\sum_{i<j} h_i^a h_j^a \omega_{ij}, \end{aligned} \tag{A.3}$$

where $h_i^a = h_i - \beta_p w_i$ is the active holding, w_i is the market weight of the share, q is the number of shares in the portfolio, Q is the number of shares in the market, $\omega_i^2 = \mathrm{var}(\epsilon_i)$, $\omega_{ij} = \mathrm{cov}(\epsilon_i, \epsilon_j)$, $\sigma_i^2 = \mathrm{var}(r_i)$, and $\sigma_{ij} = \mathrm{cov}(\epsilon_i, \epsilon_j)$. Note that $h_i^a = -\beta_p w_i$, if share i is not in the current portfolio.

Proof Because $\mathrm{E}(\epsilon_i) = 0$ for all i, then $\mathrm{E}(\epsilon_p) = \sum_{i=1}^{q} \mathrm{E}(\epsilon_i) = 0$. Hence

$$\begin{aligned} \alpha_p &= \mathrm{E}(r_p - \beta_p r_m - \epsilon_p) \\ &= \mathrm{E}(\sum_{i=1}^{q} h_i r_i - \beta_p \sum_{i=1}^{Q} w_i r_i) \\ &= \mathrm{E}(\sum_{i=1}^{Q} h_i^a r_i), \end{aligned} \tag{A.4}$$

which proves the equality between the left and rightmost formulas of (A.2).

On the other hand, since $r_i = \alpha_i + \beta_i r_m + \epsilon_i$, with $\mathrm{E}(\epsilon_i) = 0$, we can write in (A.4)

$$\alpha_p = \sum_{i=1}^{Q} h_i^a \alpha_i + \sum_{i=1}^{Q} h_i^a \beta_i \mu_m, \tag{A.5}$$

where $\mu_m = \mathrm{E}(r_m)$. But $\sum_{i=1}^{Q} h_i^a \beta_i = \sum_{i=1}^{q} h_i \beta_i - \beta_p \sum_{i=1}^{Q} w_i \beta_i = \beta_p - \beta_p = 0$, since $\sum_{i=1}^{Q} w_i \beta_i = 1$ (market beta). Hence, the last sum disappears from (A.5). Thus the proof of (A.2) is complete.

To prove relation (A.3) we write

$$\begin{aligned} \mathrm{var}(\epsilon_p) &= \mathrm{var}(r_p - \alpha_p - \beta_p r_m) \\ &= \mathrm{var}(r_p - \beta_p r_m) \qquad (\alpha_p \text{ is a constant}) \\ &= \mathrm{var}(\sum_{i=1}^{Q} h_i^a r_i) \\ &= \sum_{i=1}^{Q} (h_i^a)^2 \sigma_i^2 + 2 \sum_{i<j} h_i^a h_j^a \sigma_{ij}, \end{aligned} \tag{A.6}$$

where $\sigma_i^2 = \mathrm{var}(r_i)$ and $\sigma_{ij} = \mathrm{cov}(r_i, r_j)$. Hence, the proof of the first equality of (A.3) is complete. The second equality follows directly by replacing r_i by the market model representation. Consequently $\sum_{i=1}^{Q} h_i^a r_i = \alpha_p + \sum_{i=1}^{Q} h_i^a \epsilon_i$ (see above). Using this in the variance equation gives the last equality of (A.3), completing the proof of the proposition. □