Financial Time Series Analysis and Econometrics II

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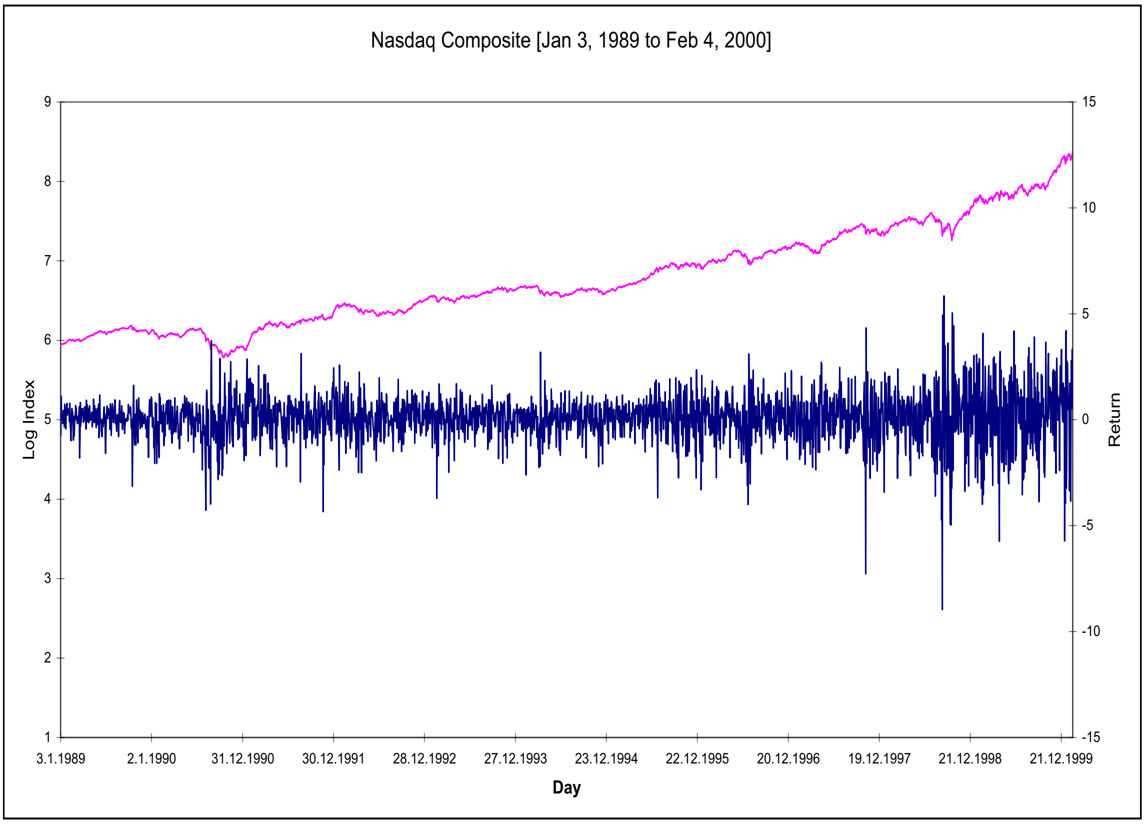
1. Nonlinear Univariate Times Series

1.1 Background

Example 1.1: Consider the following daily close-to-close Nasdaq composite share index values [January 3, 1989 to February 4, 2000]



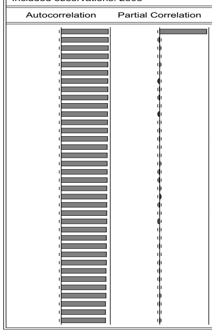
Below are autocorrelations of the log-index.

Obviously the persistence of autocorrelations indicate that the series is integrated (see definition below).

The autocorrelations of the return series suggest that the returns are stationary with statistically significant first order autocorrelation.

Correlogram of LNSDQ

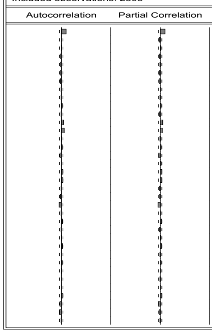
Date: 04/01/01 Time: 14:19
Sample: 2276 5080
Included observations: 2805



	AC	PAC	Q-Stat	Prob
1	0.998	0.998	2798.4	0.000
2	0.997	-0.010	5587.9	0.000
3	0.995	0.016	8369.0	0.000
4	0.993	0.001	11142.	0.000
5	0.992	0.013	13906.	0.000
6	0.990	0.006	16663.	0.000
7	0.988	-0.018	19411.	0.000
8	0.987	-0.001	22151.	0.000
9	0.985	-0.011	24882.	0.000
10	0.983	0.007	27605.	0.000
11	0.981	-0.019	30319.	0.000
12	0.980	0.003	33025.	0.000
13	0.978	-0.001	35722.	0.000
14	0.976	-0.003	38410.	0.000
15	0.974	0.005	41090.	0.000
16	0.973	0.007	43762.	0.000
17	0.971	0.016	46426.	0.000
18	0.970	-0.009	49081.	0.000
19	0.968	-0.016	51728.	0.000
20	0.966	0.014	54367.	0.000
21	0.965	0.018	56998.	0.000
22	0.963	-0.016	59621.	0.000
23	0.961	0.001	62236.	0.000
24	0.959	-0.025	64842.	0.000
25	0.958	0.005	67440.	0.000
26	0.956	0.001	70030.	0.000
27	0.954	-0.002	72611.	0.000
28	0.953	0.002	75185.	0.000
29	0.951	-0.003	77750.	0.000
30	0.949	-0.004	80306.	0.000
31	0.948	0.000	82855.	0.000
32	0.946	0.003	85395.	0.000
33	0.944	0.013	87927.	0.000
34	0.943	0.000	90451.	0.000
35	0.941	0.007	92967.	0.000
36	0.939	0.014	95476.	0.000

Correlogram of DNSDQ

Date: 04/01/01 Time: 14:19
Sample: 2276 5080
Included observations: 2805



	AC	PAC	Q-Stat	Prob
1	0.106	0.106	31.459	0.000
2	0.001	-0.010	31.465	0.000
3	0.008	0.009	31.659	0.000
4	-0.009	-0.011	31.904	0.000
5	-0.009	-0.007	32.123	0.000
6	-0.012	-0.010	32.514	0.000
7	-0.024	-0.021	34.069	0.000
8	-0.001	0.004	34.071	0.000
9	-0.006	-0.007	34.171	0.000
10	0.017	0.019	35.025	0.000
11	-0.004	-0.009	35.079	0.000
12	0.049	0.051	41.844	0.000
13	0.058	0.048	51.463	0.000
14	0.009	-0.002	51.686	0.000
15	0.017	0.016	52.507	0.000
16	-0.028	-0.032	54.694	0.000
17	0.008	0.017	54.875	0.000
18	0.036	0.034	58.504	0.000
19	0.040	0.038	63.052	0.000
20	-0.007	-0.013	63.174	0.000
21	-0.019	-0.017	64.219	0.000
22	-0.046	-0.044	70.298	0.000
23	-0.005	0.003	70.379	0.000
24	0.027	0.029	72.511	0.000
25	-0.002	-0.012	72.527	0.000
26	0.013	0.014	73.018	0.000
27	0.032	0.025	75.957	0.000
28	0.002	-0.005	75.968	0.000
29	0.020	0.021	77.127	0.000
30	0.014	0.006	77.648	0.000
31	0.005	-0.002	77.714	0.000
32	0.004	0.000	77.767	0.000
33	0.037	0.041	81.689	0.000
34	-0.032	-0.034	84.586	0.000
35	-0.037	-0.022	88.467	0.000
36	-0.006	-0.005	88.571	0.000

Figure. Nasdaq Composite index autocorrelations for log levels and log differences (returns)

Definition 1.1: Time series y_t, $t = 1, \ldots, T$ is covariance stationary if

$$\begin{aligned} \text{E}[y_t] &= \mu, \text{ for all } t \\ \text{Cov}[y_t, y_{t+k}] &= \gamma_k, \text{ for all } t \\ \text{Var}[y_t] &= \gamma_0 \ (< \infty), \text{ for all } t \end{aligned} \quad (1)$$

Any series that are not stationary are said to be nonstationary.

Definition 1.2: Time series u_t is a white noise process if

$$\begin{aligned} \text{E}[u_t] &= \mu, \text{ for all } t \\ \text{Cov}[u_t, u_s] &= 0, \text{ for all } t \neq s \\ \text{Var}[u_t] &= \sigma_u^2 < \infty, \text{ for all } t. \end{aligned} \quad (2)$$

We denote $u_t \sim WN(\mu, \sigma_u^2)$.

Remark 1.1: Usually it is assumed in (2) that $\mu = 0$.

Remark 1.2: A WN-process is obviously stationary.

Definition 1.3: Times series y_t is said to be integrated of order 1, if it is of the form

(3) $$(1 - L)y_t = \delta + \psi(L)u_t,$$

denoted as $y_t \sim I(1)$, where

(4) $$\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \cdots$$

such that $\sum_{j=1}^{\infty} |\psi_j| < \infty$, $\psi(1) \neq 0$, roots of $\psi(z) = 0$ are outside the unit circle [or the polynomial (4) is of order zero], and u_t is a white noise series with mean zero and variance σ_u^2.

Remark 1.3: If a time series process is of the form of the right hand side of (4), i.e.,

$$x_t = \delta + \psi(L)u_t, \tag{5}$$

where $\psi(L)$ satisfies the conditions of Def 1.3, it can be shown that x_t is stationary. We denote $x_t \sim I(0)$, i.e, integrated of order zero.

Remark 1.4: The assumption $\psi(1) \neq 0$ is important. It rules out for example trend stationary series

$$y_t = \alpha + \beta t + \psi(L)u_t. \tag{6}$$

Because $\text{E}[y_t] = \alpha + \beta t$, y_t is nonstationary. However,

$$(1 - L)y_t = \beta + \tilde{\psi}(L)u_t, \tag{7}$$

where

$$\tilde{\psi}(L) = (1 - L)\psi(L). \tag{8}$$

Now, although, $(1 - L)y_t$ is stationary, however,

$$\tilde{\psi}(1) = (1 - 1)\psi(1) = 0,$$

which does not satisfy the rule in Definition 1.3, and hence a trend stationary series is not $I(1)$.

Example 1.2: (Example 1.1 continued) Below are results after fitting an AR(1) and an MA(1) model to the return series

Table. AR(1) estimates.

Dependent Variable: DNSDQ
Method: Least Squares
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 2 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.086126	0.023048	3.736845	0.0002
AR(1)	0.105933	0.018782	5.640001	0.0000

R-squared	0.011221	Mean dependent var	0.086119
Adjusted R-squared	0.010868	S.D. dependent var	1.097336
S.E. of regression	1.091357	Akaike info criterion	3.013434
Sum squared resid	3338.542	Schwarz criterion	3.017668
Log likelihood	-4224.341	F-statistic	31.80961
Durbin-Watson stat	1.997947	Prob(F-statistic)	0.000000
Inverted AR Roots	.11		

Table. MA(1) estimates

Dependent Variable: DNSDQ
Method: Least Squares
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 4 iterations

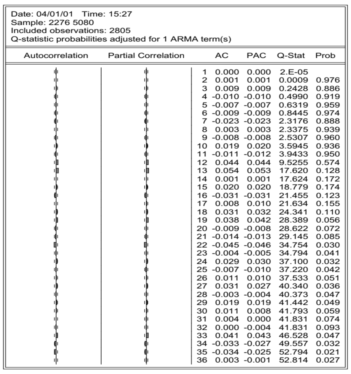
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.086153	0.022811	3.776796	0.0002
MA(1)	0.107093	0.018779	5.702685	0.0000

R-squared	0.011323	Mean dependent var	0.086119
Adjusted R-squared	0.010970	S.D. dependent var	1.097336
S.E. of regression	1.091301	Akaike info criterion	3.013331
Sum squared resid	3338.198	Schwarz criterion	3.017565
Log likelihood	-4224.196	F-statistic	32.10153
Durbin-Watson stat	2.000051	Prob(F-statistic)	0.000000
Inverted MA Roots	-.11		

Both models give virtually equally good fit, MA(1) is just only marginally better.

The residual autocorrelations and related Q-statistics indicate no further autocorrelation left to the series.

Correlogram of Residuals

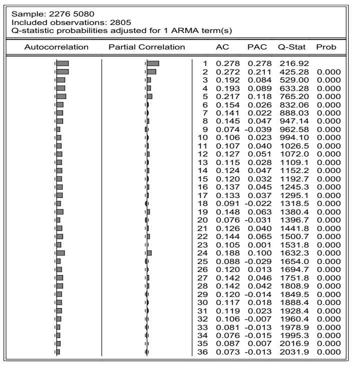


Date: 04/01/01 Time: 15:27
Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.000	0.000	2.E-05	
2	0.001	0.001	0.0009	0.976
3	0.009	0.009	0.2428	0.886
4	-0.010	-0.010	0.4990	0.919
5	-0.007	-0.007	0.6319	0.959
6	-0.009	-0.009	0.8445	0.974
7	-0.023	-0.023	2.3176	0.888
8	0.003	0.003	2.3375	0.939
9	-0.008	-0.008	2.5307	0.960
10	0.019	0.020	3.5945	0.936
11	-0.011	-0.012	3.9433	0.950
12	0.044	0.044	9.5255	0.574
13	0.054	0.053	17.620	0.128
14	0.001	0.001	17.624	0.172
15	0.020	0.020	18.779	0.174
16	-0.031	-0.031	21.455	0.123
17	0.008	0.010	21.634	0.155
18	0.031	0.032	24.341	0.110
19	0.038	0.042	28.389	0.056
20	-0.009	-0.008	28.622	0.072
21	-0.014	-0.013	29.145	0.085
22	-0.045	-0.046	34.754	0.030
23	-0.004	-0.005	34.794	0.041
24	0.029	0.030	37.100	0.032
25	-0.007	-0.010	37.220	0.042
26	0.011	0.010	37.533	0.051
27	0.031	0.027	40.340	0.036
28	-0.003	-0.004	40.373	0.047
29	0.019	0.019	41.442	0.049
30	0.011	0.008	41.793	0.059
31	0.004	0.000	41.831	0.074
32	0.000	-0.004	41.831	0.093
33	0.041	0.043	46.528	0.047
34	-0.033	-0.027	49.557	0.032
35	-0.034	-0.025	52.794	0.021
36	0.003	-0.001	52.814	0.027

Figure. Autocorrelations of the squared MA(1) residuals

Correlogram of Residuals Squared



Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.278	0.278	216.92	
2	0.272	0.211	425.28	0.000
3	0.192	0.084	529.00	0.000
4	0.193	0.089	633.28	0.000
5	0.217	0.118	765.20	0.000
6	0.154	0.026	832.06	0.000
7	0.141	0.022	888.03	0.000
8	0.145	0.047	947.14	0.000
9	0.074	-0.039	962.58	0.000
10	0.106	0.023	994.10	0.000
11	0.107	0.040	1026.5	0.000
12	0.127	0.051	1072.0	0.000
13	0.115	0.028	1109.1	0.000
14	0.124	0.047	1152.2	0.000
15	0.120	0.032	1192.7	0.000
16	0.137	0.045	1245.3	0.000
17	0.133	0.037	1295.1	0.000
18	0.091	-0.022	1318.5	0.000
19	0.148	0.063	1380.4	0.000
20	0.076	-0.031	1396.7	0.000
21	0.126	0.040	1441.8	0.000
22	0.144	0.065	1500.7	0.000
23	0.105	0.001	1531.8	0.000
24	0.188	0.100	1632.3	0.000
25	0.088	-0.029	1654.0	0.000
26	0.120	0.013	1694.7	0.000
27	0.142	0.046	1751.8	0.000
28	0.142	0.042	1808.9	0.000
29	0.120	-0.014	1849.5	0.000
30	0.117	0.018	1888.4	0.000
31	0.119	0.023	1928.4	0.000
32	0.106	-0.007	1960.4	0.000
33	0.081	-0.013	1978.9	0.000
34	0.076	-0.015	1995.3	0.000
35	0.087	0.007	2016.9	0.000
36	0.073	-0.013	2031.9	0.000

Figure. Autocorrelations of the squared MA(1) residuals

The autocorrelations of the squared residuals strongly suggest that there is still left time dependency into the series.

The dependency, however, is nonlinear by nature.

Because squared observations are the building blocks of the variance of the series, the results suggest that the variation (volatility) of the series is time dependent.

This leads to the so called ARCH-family of models.[‡]

[‡]The inventor of this modeling approach is Robert F. Engle (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50, 987–1008.

1.2. Conditional Heteroscedasticity

ARCH-models

The general setup for ARCH models is

(9) $$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t$$

with $\mathbf{x}_t = (x_{1t}, x_{2t}, \ldots, x_{pt})'$, $\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$, $t = 1, \ldots, T$, and

(10) $$u_t|\mathcal{F}_{t-1} \sim N(0, h_t),$$

where $\mathcal{F}_t$ is the information available at time t (usually the past values of u_t; $u_1, \ldots, u_{t-1}$), and

(11) $$h_t = \mathrm{Var}(u_t|\mathcal{F}_{t-1}) = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2.$$

Furthermore, it is assumed that $\omega > 0$, $\alpha_i \geq 0$ for all i and $\alpha_1 + \cdots + \alpha_q < 1$.

For short it is denoted $u_t \sim$ ARCH(q).

This reminds essentially an AR(q) process for the squared residuals, because defining $\nu_t = u_t^2 - h_t$, we can write

$$u_t^2 = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 + \nu_t. \quad (12)$$

Nevertheless, the error term ν_t is time heteroscedastic, which implies that the conventional estimation procedure used in AR-estimation does not produce optimal results here.

Properties of ARCH-processes

Consider (for the sake of simplicity) ARCH(1) process

$$h_t = \omega + \alpha u_{t-1}^2 \tag{13}$$

with $\omega > 0$ and $0 \leq \alpha < 1$ and $u_t | u_{t-1} \sim N(0, h_t)$.

(a) u_t is white noise

(i) Constant mean (zero):

$$\mathrm{E}[u_t] = \mathrm{E}[\underbrace{\mathrm{E}_{t-1}[u_t]}_{=0}] = \mathrm{E}[0] = 0. \tag{14}$$

Note $\mathrm{E}_{t-1}[u_t] = \mathrm{E}[u_t | \mathcal{F}_{t-1}]$, the conditional expectation given information up to time $t-1$.[§]

[§] The law of iterated expectations: Consider time points $t_1 < t_2$ such that $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$, then for any $t > t_2$

$$\mathrm{E}_{t_1}\left[\mathrm{E}_{t_2}[u_t]\right] = \mathrm{E}\left[\mathrm{E}[u_t | \mathcal{F}_{t_2}] | \mathcal{F}_{t_1}\right] = \mathrm{E}[u_t | \mathcal{F}_{t_1}] = \mathrm{E}_{t_1}[u_t]. \tag{15}$$

(ii) Constant variance: Using again the law of iterated expectations, we get

$$
\begin{aligned}
\mathrm{Var}[u_t] &= \mathrm{E}[u_t^2] = \mathrm{E}\left[\mathrm{E}_{t-1}[u_t^2]\right] \\
&= \mathrm{E}[h_t] = \mathrm{E}[\omega + \alpha u_{t-1}^2] \\
&= \omega + \alpha \mathrm{E}[u_{t-1}^2] \\
&\vdots \\
&= \omega(1 + \alpha + \alpha^2 + \cdots + \alpha^n) \\
&\quad + \underbrace{\alpha^{n+1}\mathrm{E}[u_{t-n-1}^2]}_{\to 0, \text{ as } n\to\infty} \\
&= \omega\left(\lim_{n\to\infty} \sum_{i=0}^{n} \alpha^i\right) \\
&= \frac{\omega}{1-\alpha}.
\end{aligned} \tag{16}
$$

(iii) Autocovariances: Exercise, show that autocovariances are zero, i.e., $E[u_t u_{t+k}] = 0$ for all $k \neq 0$. (*Hint*: use the law of iterated expectations.)

(b) The unconditional distribution of u_t is symmetric, but nonnormal:

(i) Skewness: Exercise, show that $\mathrm{E}[u_t^3] = 0$.

(ii) Kurtosis: Exercise, show that under the assumption $u_t | u_{t-1} \sim N(0, h_t)$, and that $\alpha < \sqrt{1/3}$, the kurtosis

$$\mathrm{E}[u_t^4] = 3\frac{\omega^2}{(1-\alpha)^2} \cdot \frac{1-\alpha^2}{1-3\alpha^2}. \tag{17}$$

Hint: If $X \sim N(0, \sigma^2)$ then $\mathrm{E}[(X-\mu)^4] = 3(\sigma^2)^2 = 3\sigma^4$.

Because $(1-\alpha^2)/(1-3\alpha^2) > 1$ we have that

$$(18) \quad \mathrm{E}[u_t^4] > 3\frac{\omega^2}{(1-\alpha)^2} = 3[\mathrm{Var}(u_t)]^2,$$

we find that the kurtosis of the unconditional distribution exceed that what it would be, if u_t were normally distributed.

Thus the unconditional distribution of u_t is nonnormal and has fatter tails than a normal distribution with variance equal to $\mathrm{Var}[u_t] = \omega/(1-\alpha)$.

(c) Standardized variables:

Write

(19) $$z_t = \frac{u_t}{\sqrt{h_t}}$$

then $z_t \sim$ NID$(0, 1)$, i.e., normally and independently distributed.

Thus we can always write

(20) $$u_t = z_t\sqrt{h_t},$$

where z_t independent standard normal random variables (strict white noise).

This gives us a useful device to check after fitting an ARCH model the adequacy of the specification: Check the autocorrelations of the squared standardized series.

Estimation of ARCH models

Given the model

(21) $$y_t = \mathbf{x}_t'\beta + u_t$$

with $u_t|\mathcal{F}_{t-1} \sim N(0, h_t)$, we have $y_t|\{\mathbf{x}_t, \mathcal{F}_{t-1}\} \sim N(\mathbf{x}_t'\beta, h_t)$, $t = 1, \ldots, T$.

Then the log-likelihood function becomes

(22) $$\ell(\theta) = \sum_{t=1}^{T} \ell_t(\theta)$$

with

(23) $$\ell_t(\theta) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log h_t - \frac{1}{2}(y_t - \mathbf{x}_t'\beta)^2/h_t,$$

where $\theta = (\beta', \omega, \alpha)'$.

The maximum likelihood (ML) estimate $\hat{\theta}$ is the value maximizing the likelihood function, i.e.,

$$\ell(\hat{\theta}) = \max_{\theta} \ell(\theta). \tag{24}$$

The maximization is accomplished by numerical methods.

Remark 1.5: OLS estimates of the regression parameters are inefficient (unreliable) compared to the ML estimates.

Generalized ARCH models

In practice the ARCH needs fairly many lags.

Usually far less lags are needed by modifying the model to

(25) $$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1},$$

with $\omega > 0$, $\alpha > 0$, $\delta \geq 0$, and $\alpha + \delta < 1$.

The model is called the Generalized ARCH (GARCH) model.

Usually the above GARCH(1,1) is adequate in practice.

Econometric packages call α (coefficient of u_{t-1}^2) the ARCH parameter and δ (coefficient of h_{t-1}) the GARCH parameter.

Note again that defining $\nu_t = u_t^2 - h_t$, we can write

(26) $$u_t^2 = \omega + (\alpha + \delta)u_{t-1}^2 + \nu_t - \delta\nu_{t-1}$$

a heteroscedastic ARMA(1,1) process.

Applying backward substitution, one easily gets

(27) $$h_t = \frac{\omega}{1-\delta} + \alpha \sum_{j=1}^{\infty} \delta^{j-1} u_{t-j}^2$$

an ARCH(∞) process.

Thus the GARCH term captures all the history from $t-2$ backwards of the shocks u_t.

Imposing additional lag terms, the model can be extended to GARCH(r, q) model

$$h_t = \omega + \sum_{j=1}^{r} \delta_j h_{t-j} + \sum_{i=1}^{q} \alpha u_{t-i}^2 \tag{28}$$

[c.f. ARMA(p, q)].

Nevertheless, as noted above, in practice GARCH(1,1) is adequate.

Example 1.3: MA(1)-GARCH(1,1) model of Nasdaq returns. The model is

$$\begin{aligned} r_t &= \mu + u_t + \theta u_{t-1} \\ h_t &= \omega + \alpha u_{t-1}^2 + \delta h_{t-1}. \end{aligned} \tag{29}$$

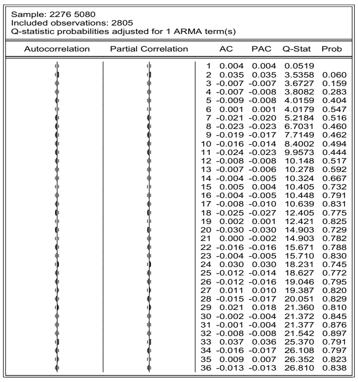
Estimation results (EViews 4.0)

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 21 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.084907	0.017700	4.797124	0.0000
MA(1)	0.171620	0.020952	8.190983	0.0000
		Variance Equation		
C	0.027892	0.009213	3.027258	0.0000
ARCH(1)	0.121770	0.020448	5.955103	0.0000
GARCH(1)	0.857095	0.021526	39.81666	0.0000

R-squared	0.007104	Mean dependent var	0.086119
Adjusted R-squared	0.005685	S.D. dependent var	1.097336
S.E. of regression	1.094213	Akaike info criterion	2.695856
Sum squared resid	3352.444	Schwarz criterion	2.706443
Log likelihood	-3775.938	F-statistic	5.008069
Durbin-Watson stat	2.129878	Prob(F-statistic)	0.000507
Inverted MA Roots	-.17		

Correlogram of Standardized Residuals Squared



Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.004	0.004	0.0519	
		2	0.035	0.035	3.5358	0.060
		3	-0.007	-0.007	3.6727	0.159
		4	-0.007	-0.008	3.8082	0.283
		5	-0.009	-0.008	4.0159	0.404
		6	0.001	0.001	4.0179	0.547
		7	-0.021	-0.020	5.2184	0.516
		8	-0.023	-0.023	6.7031	0.460
		9	-0.019	-0.017	7.7149	0.462
		10	-0.016	-0.014	8.4002	0.494
		11	-0.024	-0.023	9.9573	0.444
		12	-0.008	-0.008	10.148	0.517
		13	-0.007	-0.006	10.278	0.592
		14	-0.004	-0.005	10.324	0.667
		15	0.005	0.004	10.405	0.732
		16	-0.004	-0.005	10.448	0.791
		17	-0.008	-0.010	10.639	0.831
		18	-0.025	-0.027	12.405	0.775
		19	0.002	0.001	12.421	0.825
		20	-0.030	-0.030	14.903	0.729
		21	0.000	-0.002	14.903	0.782
		22	-0.016	-0.016	15.671	0.788
		23	-0.004	-0.005	15.710	0.830
		24	0.030	0.030	18.231	0.745
		25	-0.012	-0.014	18.627	0.772
		26	-0.012	-0.016	19.046	0.795
		27	0.011	0.010	19.387	0.820
		28	-0.015	-0.017	20.051	0.829
		29	0.021	0.018	21.360	0.810
		30	-0.002	-0.004	21.372	0.845
		31	-0.001	-0.004	21.377	0.876
		32	-0.008	-0.008	21.542	0.897
		33	0.037	0.036	25.370	0.791
		34	-0.016	-0.017	26.108	0.797
		35	0.009	0.007	26.352	0.823
		36	-0.013	-0.013	26.810	0.838

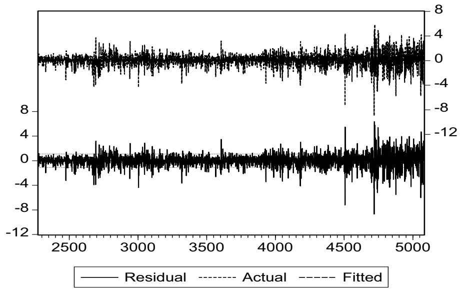


Figure. Conditional standard deviation function

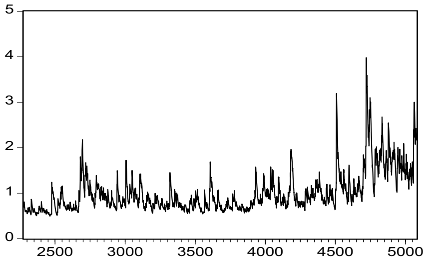
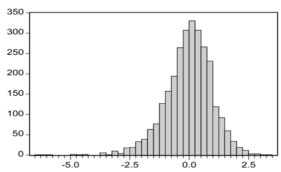


Figure. Conditional standard deviation function



Series: Standardized Residuals
Sample 2276 5080
Observations 2805

Mean	-0.019280
Median	0.050222
Maximum	3.265924
Minimum	-6.466048
Std. Dev.	0.999654
Skewness	-0.731417
Kurtosis	5.456912
Jarque-Bera	955.6050
Probability	0.000000

Figure. Conditional standard deviation function

The autocorrelations of the squared standardized residuals pass the white noise test.

Nevertheless, the normality of the standardized residuals is strongly rejected.

This is why robust standard errors are used in the estimation of the standard errors.

The variance function can be extended by including regressors (exogenous or predetermined variables), x_t, in it

$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1} + \pi x_t. \quad (30)$$

Note that if x_t can assume negative values, it may be desirable to introduce absolute values $|x_t|$ in place of x_t in the conditional variance function.

For example, with daily data a Monday dummy could be introduced into the model to capture the non-trading over the weekends in the volatility.

ARCH-M Model

The regression equation may be extended by introducing the variance function into the equation

$$y_t = \mathbf{x}_t'\beta + \gamma g(h_t) + u_t, \tag{31}$$

where $u_t \sim$ GARCH, and g is a suitable function (usually square root or logarithm).

This is called the ARCH in Mean (ARCH-M) model [Engle, Lilien and Robbins (1987)[¶]].

The ARCH-M model is often used in finance, where the expected return on an asset is related to the expected asset risk.

The coefficient γ reflects the risk-return tradeoff.

[¶] *Econometrica*, 55, 391–407.

Example 1.4: Does the daily mean return of Nasdaq depend on the volatility level?

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 22 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.198064	0.074141	2.671456	0.0076
C	-0.069416	0.061432	-1.129969	0.2585
MA(1)	0.174785	0.020644	8.466806	0.0000
		Variance Equation		
C	0.031799	0.009301	3.419007	0.0006
ARCH(1)	0.134070	0.020974	6.392287	0.0000
GARCH(1)	0.842134	0.021350	39.44407	0.0000

R-squared	0.011379	Mean dependent var	0.086119
Adjusted R-squared	0.009613	S.D. dependent var	1.097336
S.E. of regression	1.092049	Akaike info criterion	2.694709
Sum squared resid	3338.007	Schwarz criterion	2.707413
Log likelihood	-3773.330	F-statistic	6.443432
Durbin-Watson stat	2.127550	Prob(F-statistic)	0.000006
Inverted MA Roots	-.17		

The volatility term in the mean equation is statistically significant indicating that rather than being constant the mean return is dependent on the level of volatility.

Consequently, the data suggests that the best fitting model so far is of the form

$$r_t = \gamma\sqrt{h_t} + u_t + \theta u_{t-1}$$
$$h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1}.$$

Below are the estimation results for the above model

Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 16 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.119204	0.022944	5.195479	0.0000
MA(1)	0.174104	0.020771	8.382098	0.0000
	Variance Equation			
C	0.031291	0.009545	3.278211	0.0010
ARCH(1)	0.133713	0.021011	6.363810	0.0000
GARCH(1)	0.843131	0.021785	38.70279	0.0000

R-squared	0.010861	Mean dependent var	0.086119
Adjusted R-squared	0.009448	S.D. dependent var	1.097336
S.E. of regression	1.092141	Akaike info criterion	2.694578
Sum squared resid	3339.759	Schwarz criterion	2.705165
Log likelihood	-3774.146	Durbin-Watson stat	2.132844
Inverted MA Roots	-.17		

Correlogram of Standardized Residuals Squared

Sample: 2276 5080
Included observations: 2805
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.003	-0.003	0.0213	
		2	0.034	0.034	3.2371	0.072
		3	-0.008	-0.008	3.4198	0.181
		4	-0.009	-0.010	3.6300	0.304
		5	-0.009	-0.009	3.8611	0.425
		6	0.001	0.001	3.8636	0.569
		7	-0.022	-0.021	5.1942	0.519
		8	-0.023	-0.023	6.6228	0.469
		9	-0.018	-0.017	7.5391	0.480
		10	-0.015	-0.014	8.1998	0.514
		11	-0.023	-0.023	9.7327	0.464
		12	-0.007	-0.007	9.8585	0.543
		13	-0.006	-0.005	9.9568	0.620
		14	-0.001	-0.002	9.9594	0.697
		15	0.005	0.003	10.023	0.761
		16	-0.003	-0.005	10.055	0.816
		17	-0.006	-0.008	10.163	0.858
		18	-0.025	-0.027	11.907	0.806
		19	0.006	0.004	11.994	0.848
		20	-0.030	-0.030	14.490	0.754
		21	0.003	0.000	14.511	0.804
		22	-0.016	-0.015	15.193	0.813
		23	-0.002	-0.004	15.207	0.853
		24	0.031	0.031	17.937	0.761
		25	-0.011	-0.013	18.265	0.790
		26	-0.011	-0.015	18.602	0.816
		27	0.013	0.011	19.073	0.833
		28	-0.015	-0.016	19.737	0.842
		29	0.022	0.019	21.147	0.819
		30	-0.003	-0.004	21.173	0.853
		31	0.000	-0.002	21.173	0.882
		32	-0.006	-0.006	21.293	0.904
		33	0.041	0.040	26.022	0.763
		34	-0.016	-0.016	26.794	0.769
		35	0.009	0.006	27.013	0.797
		36	-0.013	-0.012	27.472	0.814

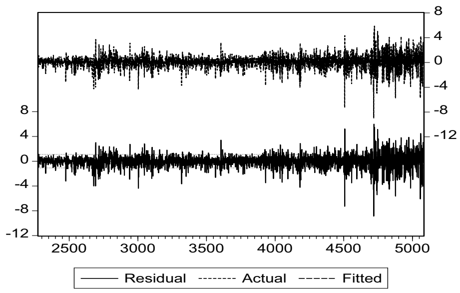
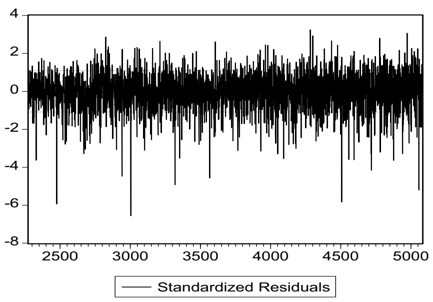
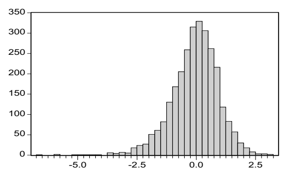


Figure. Actual and fitted series, and residuals





Series: Standardized Residuals
Sample 2276 5080
Observations 2805

Mean	-0.056142
Median	0.018376
Maximum	3.239140
Minimum	-6.557423
Std. Dev.	0.998195
Skewness	-0.726040
Kurtosis	5.409519
Jarque-Bera	924.9855
Probability	0.000000

Looking at the standardized residuals, the distribution and the sample statistics of the distribution, we observe that the residual distribution is obviously skewed in addition to the leptokurtosis.

The skewness may be due to some asymmetry in the conditional volatility which we have not yet modeled.

In financial data the asymmetry is usually such that downward shocks cause higher volatility in the near future than the positive shocks.

In finance this is called the leverage effect.

An obvious and simple first hand check for the asymmetry is to investigate the cross autocorrelations between standardized and squared standardized GARCH residuals.

Below are the cross autocorrelations between the standardized and squared standardized residuals of the fitted MA(1)-GARCH(1,1) model.

```
============================================================
    Z,Z2(-i)               Z,Z2(+i)      i     lag    lead
============================================================
    ****|         |       ****|         | 0  -0.3916 -0.3916
        |         |          *|         | 1   0.0342 -0.0782
        |         |          *|         | 2  -0.0055 -0.0842
        |         |           |         | 3   0.0093 -0.0373
        |         |           |         | 4   0.0066  0.0315
        |         |           |         | 5   0.0134 -0.0046
        |         |           |         | 6  -0.0134 -0.0019
        |         |           |         | 7   0.0113 -0.0004
        |         |           |         | 8  -0.0019  0.0045
        |         |           |         | 9  -0.0034  0.0272
        |         |           |         | 10 -0.0205  0.0128
============================================================
```

The cross autocorrelations correlations are not large, but may indicate the presence of some asymmetry.

Asymmetric ARCH: TARCH and EGARCH

A kind of stylized fact in stock markets is that downward movements are followed by higher volatility.

EViews includes two models that allow for asymmetric shocks to volatility.

The TARCH model

Threshold ARCH, TARCH (Zakoian 1994, *Journal of Economic Dynamics and Control*, 931–955 , Glosten, Jagannathan and Runkle 1993, *Journal of Finance*, 1779-1801) is given by [TARCH(1,1)]

$$(32)\quad h_t = \omega + \alpha u_{t-1}^2 + \gamma u_{t-1}^2 d_{t-1} + \delta h_{t-1},$$

where $d_t = 1$, if $u_t < 0$ (bad news) and zero otherwise.

The impact of good news is α and bad news $\alpha + \gamma$.

Thus, $\gamma \neq 0$ implies asymmetry.

Leverage exists if $\gamma > 0$.

Example 1.5: Estimation results for the MA(1)-TARCH-M model.

```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 26 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
=============================================================
                 Coefficient  Std. Error  z-Statistic  Prob.
=============================================================
SQR(GARCH)         0.091184    0.023097    3.947880    0.0001
MA(1)              0.184263    0.020899    8.816678    0.0000
=============================================================
                   Variance Equation
=============================================================
C                  0.037068     0.009513    3.896566    0.0001
ARCH(1)            0.084275     0.025080    3.360240    0.0008
(RESID<0)*ARCH(1)  0.099893     0.040881    2.443502    0.0145
GARCH(1)           0.833239     0.019001    43.85202    0.0000
=============================================================
R-squared          0.009604    Mean dependent var    0.086119
Adjusted R-squared 0.007835    S.D. dependent var    1.097336
S.E. of regression 1.093029    Akaike info criterion 2.686832
Sum squared resid  3344.000    Schwarz criterion     2.699536
Log likelihood    -3762.281    Durbin-Watson stat    2.149776
=============================================================
Inverted MA Roots         -.18
=============================================================
```

The goodness of fit improves and the statistically significant positive asymmetry parameter estimate indicates presence of leverage.

Furthermore, as seen below, the first few cross auto-correlations reduce to about one half of the original ones.

They are still statistically significant by slightly exceeding the approximate 95% boundaries $\pm 2/\sqrt{T} = \pm 2/\sqrt{2805} \approx \pm 0.038$.

Cross autocorrelations of the standardized and squared standardized MA(1)-TARCH(1,1)-M model.

```
==============================================================
   Z,Z2(-i)                Z,Z2(+i)           i      lag       lead
--------------------------------------------------------------
    ****|         |        ****|         | 0  -0.3543  -0.3543
        |         |           *|         | 1   0.0304  -0.0488
        |         |           *|         | 2  -0.0071  -0.0537
        |         |            |         | 3   0.0112  -0.0156
        |         |            |*        | 4   0.0069   0.0545
        |         |            |         | 5   0.0113   0.0080
==============================================================
```

The EGARCH model

Nelson (1991) (*Econometrica*, 347–370) proposed the Exponential GARCH (EGARCH) model for the variance function of the form (EGARCH(1,1))

$$\log h_t = \omega + \delta \log h_{t-1} + \alpha \left|z_{t-1}\right| + \gamma z_{t-1}, \tag{33}$$

where $z_t = u_t/\sqrt{h_t}$ is the standardized shock.

Again, the impact is asymmetric if $\gamma \neq 0$, and leverage is present if $\gamma < 0$.

Example 1.6: MA(1)-EGARCH(1,1)-M estimation results.

```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 28 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
=========================================================
                  Coefficient Std. Error z-Stat    Prob.
=========================================================
SQR(GARCH)           0.084631  0.022593  3.745866  0.0002
MA(1)                0.171543  0.020387  8.414399  0.0000
=========================================================
                       Variance Equation
=========================================================
C                   -0.197193  0.023051 -8.554804  0.0000
|RES|/SQR[GARCH](1)  0.251752  0.030816  8.169621  0.0000
RES/SQR[GARCH](1)   -0.071425  0.024034 -2.971755  0.0030
EGARCH(1)            0.958125  0.010941  87.57385  0.0000
=========================================================
R-squared           0.010762  Mean dependent var  0.086119
Adjusted R-squared 0.008995  S.D. dependent var  1.097336
S.E. of regression 1.092390  Akaike info criter  2.682518
Sum squared resid   3340.093  Schwarz criterion   2.695222
Log likelihood     -3756.232  Durbin-Watson stat  2.124928
=========================================================
Inverted MA Roots          -.17
=========================================================
```

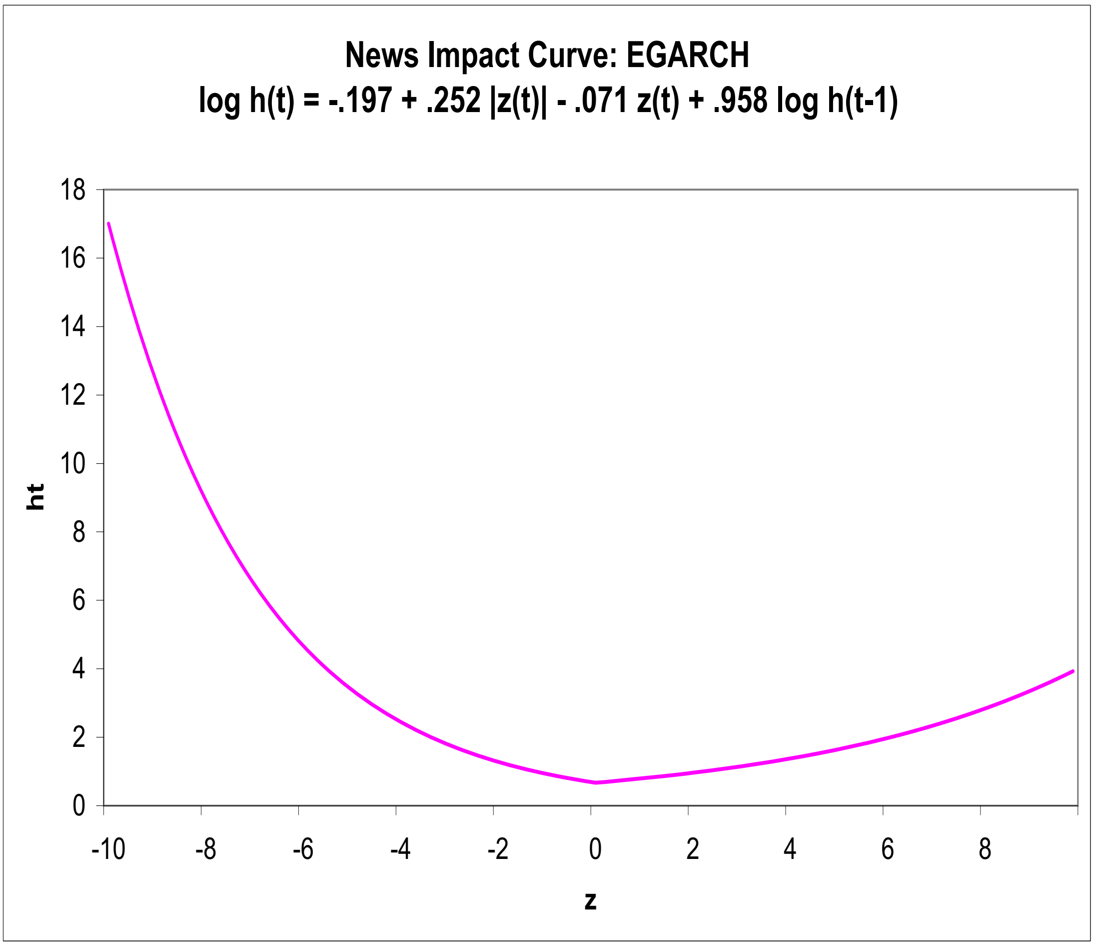
Cross autocorrelations (not shown here) are about the same as with the TARCH model (i.e., disappear). Thus TARCH and EGARCH capture most part of the leverage effect.

News Impact Curve

The asymmetry of the conditional volatility function can be conveniently illustrated by the news impact curve (NIC).

The curve is simply the graph of $h_t(z)$, where z indicates the shocks (news).

Below is a graph for the NIC of the above estimate EGARCH variance function, where h_{t-1} is replaced by the median of the estimated EGARCH series.



The Component ARCH Model

We can write the GARCH(1,1) model as

(34) $h_t = \bar{\omega} + \alpha(u_{t-1}^2 - \bar{\omega}) + \delta(h_{t-1} - \bar{\omega})$,

where

(35) $$\bar{\omega} = \frac{\omega}{1 - \alpha - \delta}$$

is the unconditional variance of the series.

Thus the usual GARCH has a mean reversion tendency towards $\bar{\omega}$.

A further extension is to allow this unconditional or long term volatility to vary over time.

This lead to so called component ARCH that allows mean reversion to a varying level q_t instead of $\bar{\omega}$.

The model is

$$
\begin{aligned}
h_t - q_t &= \alpha(u_{t-1}^2 - q_{t-1}) + \delta(h_{t-1} - q_{t-1}) \\
q_t &= \omega + \rho(q_{t-1} - \omega) + \theta(u_{t-1}^2 - h_{t-1}).
\end{aligned}
\tag{36}
$$

An asymmetric version for the model is

$$
\begin{aligned}
h_t - q_t &= \alpha(u_{t-1}^2 - q_{t-1}) \\
&\quad + \alpha(u_{t-1}^2 - q_{t-1})d_{t-1} + \delta(h_{t-1} - q_{t-1}) \\
q_t &= \omega + \rho(q_{t-1} - \omega) + \theta(u_{t-1}^2 - h_{t-1}).
\end{aligned}
\tag{37}
$$

Example 1.7: Asymmetric Component ARCH of the Nasdaq composite returns.

```
Dependent Variable: DNSDQ
Method: ML - ARCH (Marquardt)
Sample: 2276 5080
Included observations: 2805
Convergence achieved after 4 iterations
Bollerslev-Wooldrige robust standard errors & covariance
MA backcast: 2275, Variance backcast: ON
```

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.097235	0.023997	4.052002	0.0001
MA(1)	0.182908	0.027822	6.574119	0.0000
Variance Equation				
Perm: C	0.926329	0.080794	11.46533	0.0000
Perm: [Q-C]	0.734067	0.077885	9.425047	0.0000
Perm: [ARCH-GARCH]	0.228360	0.048519	4.706581	0.0000
Tran: [ARCH-Q]	0.037121	0.039565	0.938228	0.3481
Tran: (RES<0)*[ARCH-Q]	-0.077747	0.076100	-1.021635	0.3070
Tran: [GARCH-Q]	-0.688202	0.353158	-1.948710	0.0513

R-squared	0.008632	Mean dependent var	0.086119
Adjusted R-squared	0.006151	S.D. dependent var	1.097336
S.E. of regression	1.093956	Akaike info criterion	2.771137
Sum squared resid	3347.282	Schwarz criterion	2.788076
Log likelihood	-3878.519	Durbin-Watson stat	2.152828
Inverted MA Roots	-.18		

This model, however, does not fit well into the data. Thus it seems that the best fitting models so far are either the TARCH or EGARCH.

1.3 Regime switching models

A potentially useful approach to model non-linearities in time series is to assume different behavior (structural break) in different sub-samples (or regimes).

If the dates, the regimes switches have taken place, are known, modeling can be worked out simply with dummy variables.

Consider the following regression model

$$y_t = \mathbf{x}_t{}'\beta_{S_t} + u_t, \quad t = 1, \ldots, T, \tag{38}$$

where

$$u_t \sim \mathrm{NID}(0, \sigma^2_{S_t}), \tag{39}$$

$$\beta_{S_t} = \beta_0(1 - S_t) + \beta_1 S_t, \tag{40}$$

$$\sigma^2_{S_t} = \sigma^2_0(1 - S_t) + \sigma^2_1 S_t, \tag{41}$$

and

$$S_t = 0 \text{ or } 1, \quad \text{(Regime } 0 \text{ or } 1\text{)}. \tag{42}$$

Thus, under regime 1 the coefficient parameter vector is β_1 and error variance σ^2_1.

For the sake of simplicity, consider an AR(1) model.

That is, $\mathbf{x}_t = (1, y_{t-1})'$.

Usually it is assumed that the possible difference between the regimes is a mean and/or a volatility shift, but no change in the autoregression parameter.

That is,

$$(43) \quad y_t = \mu_{S_t} + \phi_1(y_{t-1} - \mu_{s_{t-1}}) + u_t,$$

with

$$(44) \quad u_t \sim \text{NID}(0, \sigma^2_{S_t}),$$

where $\mu_{S_t} = \mu_0(1 - S_t) + \mu_1 S_t$ and $\sigma^2_{S_t}$ as defined above.

If S_t, $t = 1, \ldots, T$ is known a priori, then the problem is just a usual dummy variable autoregression problem.

In practice, however, the prevailing regime is not usually directly observable.

Denote then

(45) $$P(S_t = j|S_{t-1} = i) = p_{ij},$$

$i, j = 0, 1$, called transition probabilities, with $p_{i0} + p_{i1} = 1$, $i = 0, 1$.

This kind of process, where the next state depends only on the previous state, is called the Markov process, and the model a Markov switching model in the mean and variance.

Thus, in this model additional parameters to be estimated are the transition probabilities p_{ij}.

Usually the parameters are estimated (numerically) by the ML method.**

**For a detailed discussion, see Kim Chang-Jin and Charles A. Nelson (1999). *State Space Models with Regime Switching. Classical and Gibbs-Sampling Approaches with Applications.* MIT-Press.

The joint probability density function for y_t, S_t, S_{t-1}, given past information $\mathcal{F}_{t-1} = \{y_{t-1}, y_{t-2}, \ldots\}$, is

$$f(y_t, S_t, S_{t-1}|\mathcal{F}_{t-1}) = f(y_t|S_t, S_{t-1}, \mathcal{F}_{t-1})P(S_t, S_{t-1}|\mathcal{F}_{t-1}), \tag{46}$$

with

$$f(y_t|S_t, S_{t-1}, \mathcal{F}_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2_{S_t}}} \text{exp} \left\{ -\frac{[y_t - \mu_{S_t} - \phi_1(y_{t-1} - \mu_{S_{t-1}})]^2}{2\sigma^2_{S_t}} \right\}. \tag{47}$$

Then the log-likelihood function to be maximized with respect to the unknown parameters is

$$\ell(\theta) = \sum_{t=1}^{T} \ell_t(\theta), \tag{48}$$

where

$$\ell_t(\theta) = \log \left[\sum_{S_t=0}^{1} \sum_{S_{t-1}=0}^{1} f(y_t|S_t, S_{t-1}, \mathcal{F}_{t-1}) P[S_t, S_{t-1}|\mathcal{F}_{t-1}] \right], \tag{49}$$

$$\theta = (p, q, \mu_0, \mu_1, \phi_1, \sigma_0^2, \sigma_1^2), \tag{50}$$

with

$$p = P[S_t = 0|S_{t-1} = 0], \tag{51}$$

$$q = P[S_t = 1|S_{t-1} = 1], \tag{52}$$

being the transition probabilities.

In order to evaluate the log-likelihood function we need to define the joint probabilities $P[S_t, S_{t-1}|\mathcal{F}_{t-1}]$.

Because of the Markov property

$$P[S_t|S_{t-1}, \mathcal{F}_{t-1}] = P[S_t|S_{t-1}], \quad (53)$$

we can write

$$P[S_t, S_{t-1}|\mathcal{F}_{t-1}] = P[S_t|S_{t-1}]P[S_{t-1}|\mathcal{F}_{t-1}], \quad (54)$$

and the problem reduces to calculating (estimating) the time dependent state probabilities, $P[S_{t-1}|\mathcal{F}_{t-1}]$, and weight them with the transition probabilities to obtain the joint probability.

This can be achieved as follows:

First, let $P[S_0 = 1|\mathcal{F}_0] = P[S_0 = 1] = \pi$ be given (then $P[S_0 = 0] = 1 - \pi$).

Then the probabilities $P[S_{t-1}|\mathcal{F}_{t-1}]$ and the joint probabilities are obtained using the following two steps algorithm

1^0 Given $P[S_{t-1} = i|\mathcal{F}_{t-1}]$, $i = 0, 1$, at the beginning of time t (t^{th} iteration),
$P[S_t = j, S_{t-1} = i|\mathcal{F}_{t-1}] = P[S_t = j|S_{t-1} = i]P[S_{t-1} = i|\mathcal{F}_{t-1}]$,
(55)

2^0 Once y_t is observed, we update the information set $\mathcal{F}_t = \{\mathcal{F}_{t-1}, y_t\}$ and the probabilities

$$
\begin{aligned}
P[S_t = j, S_{t-1} = i|\mathcal{F}_t] &= P[S_t = j, S_{t-1} = i|\mathcal{F}_{t-1}, y_t] \\
&= \frac{f(S_t{=}i, S_{t-1}{=}j, y_t|\mathcal{F}_{t-1})}{f(y_t|\mathcal{F}_{t-1})} \\
&= \frac{f(y_t|S_t{=}j, S_{t-1}{=}i, \mathcal{F}_{t-1})P[S_t{=}j, S_{t-1}{=}i|\mathcal{F}_{t-1}]}{\sum_{s_t, s_{t-1}=0}^{1} f(y_t|s_t, s_{t-1}, \mathcal{F}_{t-1})P[S_t{=}s_t, S_{t-1}{=}s_{t-1}|\mathcal{F}_{t-1}]}
\end{aligned}
$$

(56)
with

$$P[S_t = s_t|\mathcal{F}_t] = \sum_{s_{t-1}=0}^{1} P[S_t = s_t, S_{t-1} = s_{t-1}|\mathcal{F}_t].$$

(57)

Once we have the joint probability for the time point t, we can calculate the likelihood $\ell_t(\theta)$.

The maximum likelihood estimates for θ is then obtained iteratively maximizing the likelihood function by updating the likelihood function at each iteration with the above algorithm.

Steady state probabilities

The probabilities $\pi = P[S_0 = 1|\mathcal{F}_0]$ is called the steady state probability, and, given the transition probabilities p and q, is obtained as

(58) $$\pi = P[S_0 = 1|\mathcal{F}_0] = \frac{1-p}{2-p-q}.$$

Note that in the two state Markov chain

$$P[S_0 = 0|\mathcal{F}_0] = 1 - P[S_0 = 1|\mathcal{F}_0] = \frac{1-q}{2-p-q}.$$
(59)

Smoothed probabilities

Recall that the state S_t is unobserved.

However, once we have estimated the model, we can make inferences on S_t using all the information from the sample.

This gives us

$$P[S_t = j|\mathcal{F}_T], \quad j = 0, 1, \tag{60}$$

which are called the smoothed probabilities (for details, see Kim and Nelson 1999, pp. 68–69).

Remark 1.6: In the estimation procedure we derived $P[S_t = j|\mathcal{F}_t]$ that are usually called the filtered probabilities.

Expected duration

The expected length the system is going to stay in state j can be calculated from the transition probabilities. Let D denote the number of periods the system is in state j. The probabilities are easily found to be equal to $P[D=k] = p_{jj}^{k-1}(1-p_{jj})$, so that

$$\text{(61)} \quad \mathrm{E}[D] = \sum_{k=1}^{\infty} kP[D=k] = \frac{1}{1-p_{jj}}.$$

Note that in our case $p_{00} = p$ and $p_{11} = q$.

Example 1.8: Are there long swings in the dollar/sterling exchange rate?

If the exchange rate x_t is RW with long swings, it can be modeled as

$$\Delta x_t = \alpha_0 + \alpha_1 S_t + \epsilon_t,$$

so that $\Delta x_1 \sim N(\mu_0, \sigma_0^2)$ when $S_t = 0$ and $\Delta x_t \sim N(\mu_1, \sigma_1^2)$, when $S_t = 1$, where $\mu_0 = \alpha_0$ and $\mu_1 = \alpha_0 + \alpha_1$. Parameters μ_0 and μ_1 constitute two different drifts (if $\alpha_1 \neq 0$) in the random walk model.

Estimating the model from quarterly data for sample period 1972I to 1996IV gives

Parameter	Estimate	Std err
μ_0	2.605	0.964
μ_1	-3.277	1.582
σ_0^2	13.56	3.34
σ_1^2	20.82	4.79
p (regime 1)	0.857	0.084
q (regime 0)	0.866	0.097

The expected length of stay in regime 0 is given by $1/(1-p) = 7.0$ quarters, and in regime 1 $1/(1-q) = 7.5$ quarters.

Example 1.9: Suppose we are interested whether the market risk of a share is dependent on the level of volatility on the market. In the CAPM world the market risk of a stock is measured by β.

Consider for the sake of simplicity only the cases of high and low volatility.

The market model is

$$y_t = \alpha_{S_t} + \beta_{S_t} x_t + \epsilon_t,$$

where $\alpha_{S_t} = \alpha_0(1 - S_t) + \alpha_1 S_t$, $\beta_{S_t} = \beta_0(1 - S_t) + \beta_1 S_t$ and $\epsilon_t \sim N(0, \sigma^2_{S_t})$ with $\sigma^2_t = \sigma^2_0(1 - S_t) + \sigma^2_1 S_t$.

Estimating the model yields

Parameter	Estimate	Std Err	t-value	p-value
α_0 (low)	-0.0068	0.0178	-0.39	0.700
α_1 (high)	0.0802	0.0508	1.57	0.114
β_0 (low)	0.9679	0.0215	45.04	0.000
β_1 (high)	1.8040	0.0690	26.15	0.000
σ_0^2 (low)	0.5225	0.0198	26.37	0.000
σ_1^2 (high)	1.7050	0.0711	23.96	0.000
State Prob				
P(High|High)	0.96417			
P(Low|High)	0.03583			
P(High|Low)	0.01728			
P(Low|Low)	0.98272			
P(High)	0.67471			
P(Low)	0.32529			
Log-likelihood -3208.438				

The empirical results give evidence that the stock's market risk depends on the level of stock volatility. The expected duration of high volatility is $1/(1 - .9642) \approx 27$ days, and for low volatility 59 days.

Market returns with high-low volatility probabilities