5. An Introduction to Copulas

Dependence between random variables is indicated by the joint distributions.

Traditional assumption for stock returns is the they are time independent and follow multivariate normal distribution (i.i.d $\mathtt{MVN}(\boldsymbol{\mu}, \Sigma)$).

In such a case the dependence structure reduces to correlations available from the covariance matrix Σ.

Correlations measure linear dependencies.

Thus, correlations do not work with non-linear dependencies.

Need of knowledge of the complete joint distribution.

Deriving joint distributions are generally next to impossible without heavy assumptions.

At best one can hope to approximate marginal distributions.

Copulas are functions for 'coupling' the marginal distributions to joint distributions.

Traditional multivariate analysis: Dependence and joint distribution combined.

Copulas: Separate dependence and marginal distributions.

Let

$$F_1(x) = P[X \leq x]$$

and

$$F_2(y) = P[Y \leq y]$$

be cumulative distribution functions (CDFs) of the random variables X and Y and let $F(x, y) = P[X \leq x, Y \leq y]$ is their joint distribution.

Then mathematical theory says that under fairly general conditions there is a unique function C, such that

$$F(x, y) = C(F_1(x), F_2(y)). \tag{1}$$

C is called a *copula* function.

Thus if one knows C the joint distribution $F(x, y)$ can be derived from the marginal distributions $F_1(x)$ and $F_2(x)$

Denoting the probabilities $u = F_1(x)$ and $v = F_2(y)$, we can (usually) take $x = F_1^{-1}(u)$ and $y = F_2^{-1}(v)$.

Then

(2) $F(x, y) = F(F_1^{-1}(u), F_2^{-1}(v)) = C(u, v)$

is a copula (Sklar's theorem).

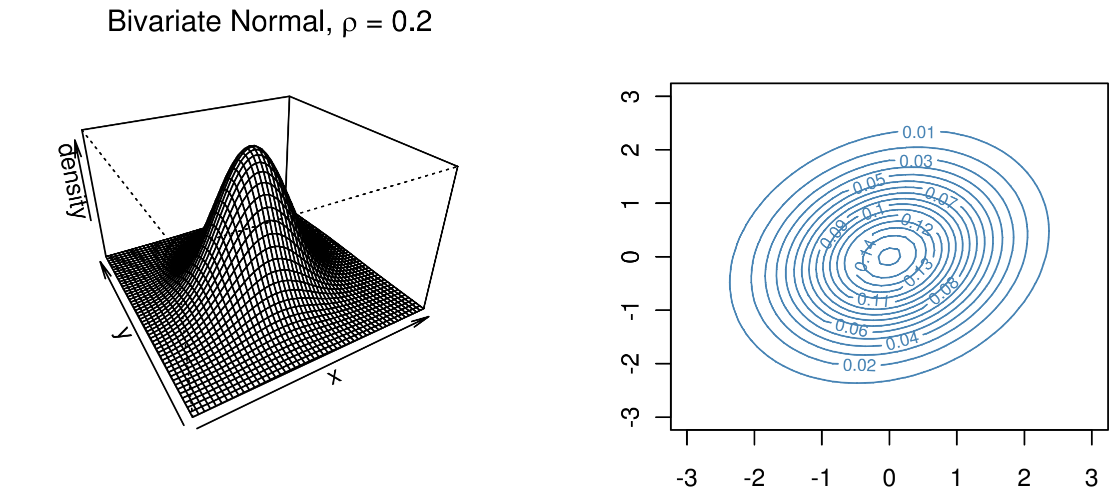
Gaussian copula:

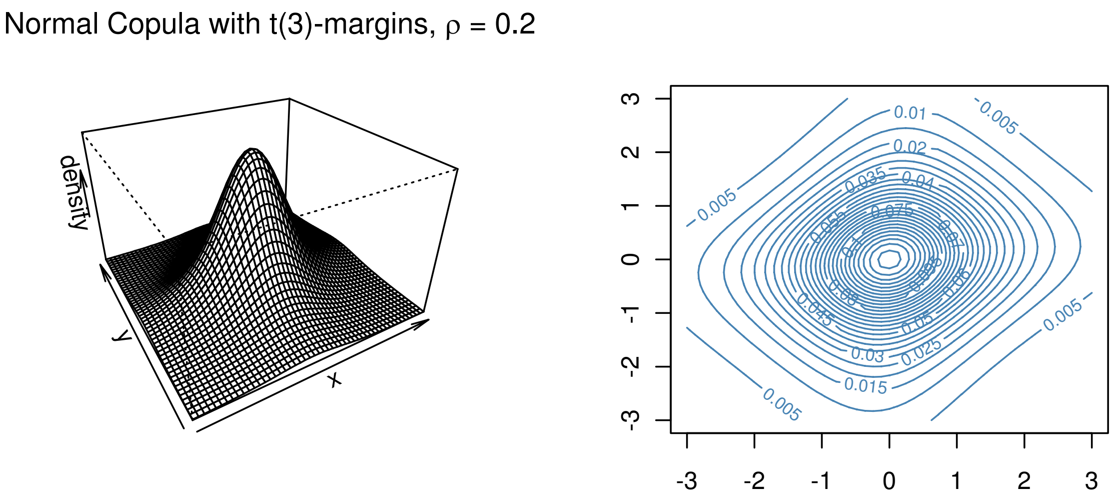
$$(3) \qquad C(u,v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right),$$

Φ_ρ is the bivariate standard normal distribution function with correlation ρ, Φ is the standard univariate normal distribution.

If the probabilities u and v are generated from normal distribution, the resulting joint distribution is normal, otherwise not.

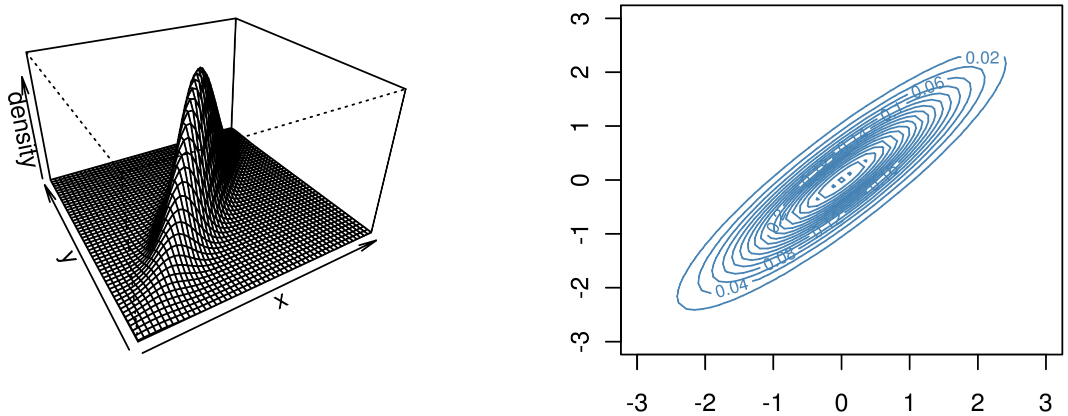
Example 5.1: Density and contour plots for bivariate normal (normal margins) $\rho = 0.2$ and normal copula ($\rho = 0.2$) with $t(\nu)$, $\nu = 3$ margins:



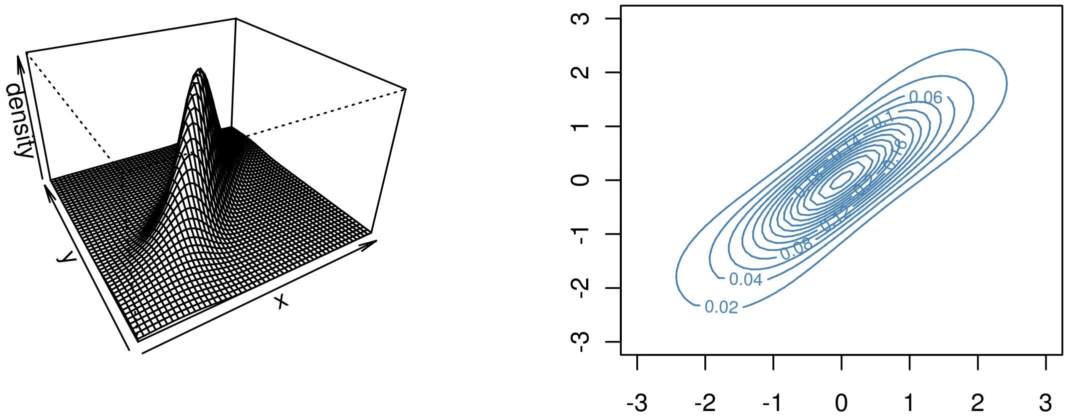


Same as above with correlation $\rho = 0.90$.

Bivariate Normal, ρ = 0.9



Normal Copula with t(3)-margins, ρ = 0.9



There are several other popular copula functions:

Examples are:

Studenst's copula

$$T_{\rho,\nu} = t_{\rho,\nu}(t_\nu^{-1}(u), t_\nu^{-1}(v)), \tag{4}$$

where $t_{\rho,\nu}$ is bivariate t distribution and t_ν is t-distribution with ν degrees of freedom. Parameter $\rho \in (-1, 1)$ is correlation.

Normal and Strudent's copulas are examples of parametric copulas.

Archimedean copulas const several types of copulas: e.g. Gumbel, Clayton, Frank, which are one parameter copulas.

Another family of copulas is the Fréchet family.

Multidimensional extensions are pretty straightforward forward.

Copulas give the joint probabilities.

Example 5.2: Bivariate digital option, which pays one unit of currency if two stocks are above (or below) ap pair of strike levels.

Consider a product on Nikkei 225 (NK225) and S&P500 indexes which pays at exercise date T one unit if NK225 is below $K_{\mathtt{NK}}$ and SP is below $K_{\mathtt{SP}}$ on the exercise date.

Given the joint CDF, the price of the put option (digital put DP) is (no-arbitrage conditions)

$$DP = e^{-r\tau} Q(K_{\mathtt{NK}}, K_{\mathtt{SP}}) \tag{5}$$

where $\tau = T - t$ and $Q(K_{\mathtt{NK}}, K_{\mathtt{SP}})$ is the joint risk-neutral probability that NK255 and SP500 are below the respective thresholds.

Suppose that we can derive the marginal CDFs $Q_{\mathtt{NK}}$ and $Q_{\mathtt{SP}}$ and we have a copula function, then

$$Q(x, y) = C(Q_{\mathtt{NK}}(x), Q_{\mathtt{SP}}(y)). \tag{6}$$

Thus,

$$\begin{aligned} DP &= e^{-r\tau} Q(K_{\mathtt{NK}}, K_{\mathtt{SP}}) \\ &= e^{-r\tau} C\left(Q_{\mathtt{NK}}(K_{\mathtt{NK}}), Q_{\mathtt{SP}}(K_{\mathtt{SP}})\right). \end{aligned} \tag{7}$$

Fréchet bounds:

$$\max(u + v - 1) \le C(u, v) \le \min(u, v). \tag{8}$$

End of Notes.