8. Intertemporal Equilibrium Models

8.1 The Stochastic Discount Factor

Random variable $M_t$ is a stochastic discount factor if

\[ 1 = E_t [(1 + R_{t+1})M_{t+1}] \]

This equation can be derived merely from the arbitrage theory, without assuming that investors maximize a well behaved utility function.

In the discrete-state setting with states $s = 1, \ldots, S$ and assets $i = 1, \ldots, N$ this goes as follows:

Let $q = (q_1, \ldots, q_N)'$ be the asset price vector, and $X_{si}$ the payoff of asset $i$ in the state $s$. The $S \times N$ payoff matrix is then

\[ X = \begin{pmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & \cdots & \vdots \\ X_{S1} & \cdots & X_{SN} \end{pmatrix}. \]
We call $p = (p_1, \ldots, p_S)'$ the state price vector if it satisfies

$$X'p = q,$$

i.e.,

$$q_i = \sum_{s=1}^{S} p_s X_{si}. \quad (2)$$

Thus $p_s$ gives the price of one dollar to be paid in state $s$.

The gross return of the $i$th asset in state $s$ is

$$g_{si} = X_{si}/q_i = 1 + R_{si}. \quad (3)$$

Dividing both sides of (2) by $q_i$ gives

$$1 = \sum_{s=1}^{S} p_s(1 + R_{si}).$$
An important result is that there exists a positive state price vector if and only if there are no arbitrage opportunities.

Let $\pi_s > 0$ denote the probability of state $s$, and define $M_s = p_s/\pi_s$, which is positive in the no arbitrage case. Then

$$1 = \sum_{s=1}^S p_s (1 + R_{si})$$

(4)

$$= \sum_{s=1}^S \pi_s M_s (1 + R_{si})$$

$$= \mathbb{E}[(1 + R_i)M],$$

which is the static discrete equivalent of (1).

**Note.** Expectation (1) holds also unconditionally, because by taking expectations from both sides yields (after reindexing)

$$1 = \mathbb{E}[(1 + R_{it})M_t].$$

(5)

Furthermore, using the definition of covariance, we have

$$\mathbb{E}[(1 + R_{it})M_t] = \mathbb{E}[1 + R_{it}]\mathbb{E}[M_t] + \text{Cov}[R_{it}, M_t],$$

from which we obtain an expression for the cross return

$$\mathbb{E}[1 + R_{it}] = \frac{1}{\mathbb{E}[M_t]} (1 + \text{Cov}[R_{it}, M_t]).$$

(6)
8.1.1 Volatility Bounds

Any model of expected returns may be viewed as a model of the stochastic discount factor.

Question: What asset return data may tell about the behavior of the stochastic discount factor.

• Lower bound for the stochastic discount factor*

Redefine

\( 1 = \mathbb{E} [(1 + \mathbf{R}_t) M_t] \),

where \( 1 = (1, \ldots, 1)' \) is an \( N \)-vector of ones, and \( \mathbf{R}_t = (R_{1t}, \ldots, R_{Nt})' \) is the return vector.

Let $M_t^*(\bar{M})$ be any stochastic discount factor with mean $\bar{M}$. Hansen and Jagannathan show that using asset pricing theory, there exist a coefficient $\beta_{\bar{M}}$ such that

$$M_t^*(\bar{M}) = \bar{M} + (R_t - E[R_t])' \beta_{\bar{M}}. \tag{8}$$

Then if $M_t^*(\bar{M})$ is a stochastic discount factor, it must satisfy (7),

$$1 = E[(1 + R_t)M_t^*(\bar{M})].$$

Expanding this yields

$$1 = \bar{M}E[1 + R_t] + \Omega \beta_{\bar{M}}, \tag{9}$$

where $\Omega = \text{Cov}(R_t)$, assumed nonsingular.

Then

$$\beta_{\bar{M}} = \Omega^{-1} (1 - \bar{M}E[1 + R_t]). \tag{10}$$
The variance of $M_t^*(\bar{M})$ becomes then

$$\text{Var}[M_t^*(\bar{M})] = \beta_M \Omega \beta_M' = (1 - \bar{M}E[1 + R_t])' \Omega (1 - \bar{M}E[1 + R_t]).$$

(11)

The right hand side of (11) is the lower bound of any discount factor with mean $\bar{M}$.

**The Benchmark Portfolio**

Let $\tilde{R}_t = (R_0, R_t')'$ be an $N+1$ vector, where $R_{0t} = 1/\bar{M} - 1$ is the return of an artificial riskless asset.

Define then the benchmark portfolio return as

$$R_{bt}(m) = \frac{M_t^*(\bar{M})}{E[M_t^*(\bar{M})^2]} - 1.$$  

(12)
**Exercise:** Show that this return can be obtained by forming a portfolio of the risky assets and the artificial riskless asset, and that it satisfies the condition (7) on returns.

**Exercise:**

(P1) Show that \( R_{bt} \) is mean-variance efficient.

(P2) Any stochastic discount factor \( M_t(\bar{M}) \) has a greater correlation with \( R_{bt} \) than with any other portfolio.

(P3) All asset returns obey a beta-pricing relation with the benchmark portfolio. That is,

\[
E \left[ R_{it} - \left( \frac{1}{M} - 1 \right) \right] = \beta_{ib} \left( E[R_{bt}] - \left( \frac{1}{M} - 1 \right) \right).
\]
Two further properties are:

\[
\frac{\sigma[R_{bt}]}{E[1 + R_{bt}]} = \frac{1/\bar{M} - E[1 + R_{bt}]}{\sigma[R_{bt}]}.
\]

(14)

\[
\frac{\sigma[1 + R_{bt}]}{E[1 + R_{bt}]} \leq \frac{\sigma[M_t(\bar{M})]}{E[M_t(\bar{M})]},
\]

(15)

i.e., the left hand side is the lower bound for the ratio of the standard deviation to mean of a stochastic discount factor.

The above analysis applies to returns themselves. Let \( Z_{it} = R_{it} - R_{kt} \) denote the excess return on asset \( i \) over asset \( k \), and let \( Z_t = (Z_{1t}, \ldots, Z_{Nt})' \). Then (7) implies

\[
0 = E[Z_t M_t].
\]

(16)
Proceeding as before, from $M_t^*(\bar{M}) = \bar{M} + (Z_t - E[Z_t])'\tilde{\beta}\bar{M}$, we get $\tilde{\beta} = \tilde{\Omega}^{-1}(-\bar{M}E[Z_t])$, where $\tilde{\Omega} = \text{Cov}[Z_t]$.

The lower bound for the stochastic discount factor is then

$$\text{(17) } \text{Var}[M^*_t(\bar{M})] = \bar{M}^2E[Z]'\tilde{\Omega}^{-1}E[Z].$$

Specially in the case of a single asset this simplifies to

$$\text{(18) } \frac{\sigma[M^*_t(\bar{M})]}{\bar{M}} = \frac{E[Z_t]}{\sigma[Z_t]}.$$

**Equity Premium**

Mehra and Prescott (1985) observe that SP-index annual standard deviation is about 18%, and mean over a commercial paper of 6%. Then the right hand side of (17) would be $6/18 = 0.33$, meaning that $\sigma(M^*_t(\bar{M})) \geq 0.33$, i.e. 33%, if $\bar{M} = 1$ (which is close to what empirically is a plausible value).
Nevertheless, empirical evidence indicates that annual standard deviation of the stochastic discount factor is much less than 33%. An implication of this would be that the equity premium cannot be explained easily with these kinds of asset pricing models.

### 8.2 Consumption-Based Asset Pricing with Power Utility Function

Let

\[ U(C_t) = \frac{C_t^{1-\gamma} - 1}{1 - \gamma}, \]

where \( \gamma \) is the relative risk aversion. Not that \( \lim_{\gamma \to 1} = \log C_t \).

*It is assumed here that \( C_t \) represents all individuals in the economy, and is the aggregate consumption. The associate asset pricing model is then Consumption CAPM or CCAPM.*

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Now

\[ U'(C_t) = C^{-\gamma}, \]

and

\[ M_{t+1} = \delta \frac{U'(C_{t+1})}{U'(C_t)} = \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \]

Implying that

\[ 1 = E_t[(1 + R_{i,t+1})M_{t+1}] \]

becomes

(20) \[ 1 = E_t \left[ (1 + R_{t+1})\delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]. \]

To test the restrictions imposed by this model we need to consider the distributional assumptions of the random variables. Assume that the joint conditional distribution of the asset returns and consumption homoscedastic and lognormal.
For a lognormally distributed random variable \( X \) holds

\[
\log E_t[X] = E_t[\log X] - \frac{1}{2} \text{Var}_t[\log X].
\]

(21)

Furthermore, the conditional homoscedasticity implies that

\[
\text{Var}_t[\log X] = \text{Var}[\log X - E_t[\log X]].
\]

Using these, and taking logs on both sides of (20) yields

\[
0 = E_t[r_{t+1} + \log \delta - \gamma E_t[\Delta c_{t+1}]
+ \frac{1}{2}[\sigma_i^2 + \gamma^2 \sigma_c^2 - 2\gamma \sigma_{ic}],
\]

(22)

where the lowercase letter denote logarithms of the corresponding variables, and

\[
\sigma_{ic} = \text{Cov} \left[ r_{it} - E_t[r_{i,t+1}], \Delta c_t - E_t[\Delta c_{t+1}] \right].
\]
The riskless (real) interest rate return is then

\[ r_{f,t+1} = -\log \delta - \frac{1}{2} \gamma^2 \sigma_c^2 + \gamma E_t[\Delta c_{t+1}] \]  

(23)

Note that we can write also

\[ E_t[\Delta c_{t+1}] = \frac{1}{2} \gamma \sigma_c^2 + \psi (r_{f,t+1} + \log \delta), \]

(24)

where \( \psi = 1/\gamma \).

Furthermore, the homoscedasticity assumption makes the log risk premium on any asset over the riskless real rate constant, so that

\[ E_t[r_{i,t+1} - r_{f,t+1}] + \frac{1}{2} \sigma_i^2 = \gamma \sigma_{ic}, \]

(25)

or remembering that e.g. \( r_{it} = \log(1 + R_{it}) \) and using (21) we can write

\[ \log E_t [(1 + R_{i,t+1})/(1 + R_{f,t+1})] = \gamma \sigma_{ic}. \]

(26)

This shows that the risk premia are determined by the coefficient of relative risk aversion times covariance with consumption growth.
### Table 8.1 Sample statistics of US consumption growth and asset returns

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Correl. with consumption growth</th>
<th>Covar with consumption growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption growth</td>
<td>0.0172</td>
<td>0.0328</td>
<td>1.0000</td>
<td>0.0011</td>
</tr>
<tr>
<td>Stock return</td>
<td>0.0601</td>
<td>0.1674</td>
<td>0.4902</td>
<td>0.0027</td>
</tr>
<tr>
<td>CP return</td>
<td>0.0183</td>
<td>0.0544</td>
<td>-0.1157</td>
<td>-0.0002</td>
</tr>
<tr>
<td>Stock-CP return</td>
<td>0.0418</td>
<td>0.1774</td>
<td>0.4979</td>
<td>0.0029</td>
</tr>
</tbody>
</table>

Using these figures in (25) we obtain $\gamma = 19$ to fit the equity premium, which is much greater than 10, the maximum value considered plausible by Mehra and Prescott!
Time Varying Expected Returns and Consumption Growth

Equation (22) gives a relation between rational expectations of asset returns and rational expectations of consumption growth.

Define an error term

\[ \eta_{i,t+1} = r_{i,t+1} - E_t[r_{i,t+1}] - \gamma(\Delta c_{t+1} - E_t[\Delta c_{t+1}]) , \]

so that we can rewrite (22)

(27) \[ r_{i,t+1} = \mu_i + \gamma \Delta c_{t+1} + \eta_{i,t+1}. \]

In general the error term \( \eta_{i,t+1} \) are correlated with realized consumption growth. So OLS is not an appropriate estimation method. However, \( \eta_{i,t+1} \) is not correlated with any information variables at time \( t \). Hence any lagged variables can be used as instruments in IV regression.
8.3 GMM Estimation in Discount Factor Models

The Generalized Method of Moments (GMM) is a natural estimation method in discount factor models (DFM). The asset pricing model predicts

$$E[P_t] = E[M(\text{data}_{t+1}, b)P_{t+1}],$$

where $b$ denote the parameters of the model. Note that $P_t$ is usually a vector of prices.

For example with the power utility function

$$M(\text{data}_{t+1}, b) = \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma},$$

so that $b = (\delta, \gamma)'$. 
Natural estimates of the left and right hand sides are the sample averages

\[
\frac{1}{T} \sum_{t=1}^{T} P_t, \tag{30}
\]

and

\[
\frac{1}{T} \sum_{t=1}^{T} [M(\text{data}_{t+1}, \text{parms})P_{t+1}] \tag{31}
\]

GMM estimates the parameters by making the sample averages (30) and (31) as close to each other as possible.
Estimation Procedure

Define **errors**

(32) \( u_{t+1}(b) = M_{t+1}(b)P_{t+1} - P_t \),

**pricing error**

(33) \( g_T(b) = \frac{1}{T} \sum_{t=1}^{T} u_t(b) \),

GMM estimate:

(34) \( \hat{b} = \text{argmin}_b g_T(b)'\hat{S}^{-1}g_T(b) \),

where \( \hat{S}^{-1} \) is an estimate of

(35) \( S = \sum_{j=-\infty}^{\infty} \mathbb{E} \left[ u_t(b)u_{t-j}(b)' \right] \),

An estimate of \( S \) is

(36) \( \hat{S} = \frac{1}{T} \sum_{t=1}^{T} u_t(\hat{b})u_t(\hat{b})' \)

In practice the estimation is worked out numerically with suitable starting values. In the first round \( \hat{S} = I \), the identity matrix.
The essential point here is that error, \( u_{t+1}(b) \) should be unpredictable.* That is

\[
(37) \quad E \left[ u_{t+1}(b) | I_t \right] = E_t \left[ u_{t+1}(b) \right] = 0,
\]

where \( I_t \) is the available information at time point \( t \). We say that the prediction errors are orthogonal (uncorrelated) to \( I_t \).

*We say that a stochastic process, \( Y_t \), is unpredictable if its conditional mean is the same as its unconditional mean. That is \( E_t [Y_{t+1}] = E [Y_t] = \mu_Y \).
Usually the information set $I_t$ consist of some predictor variables, say $z_t$. These are also called as instrumental variables. The conditions that the instrument variables are not correlated with the prediction errors are called orthogonality conditions, and are mathematically defined as

$$\mathbb{E}[u_{t+1}(b)z_t] = 0,$$

i.e., $u_{t+1}(b)$ is uncorrelated with each component of $z_t$.

**Note.** The expectation (37) is of the form (39) with $z_t = 1$.

If the number of equations (moments) in (39) is less than the number of parameters in $b$ then we say that the estimation problem is **underidentified**, and the estimates of the parameters cannot be solved uniquely.
If there are equally many moments as parameters then the problem is **exactly identified** and the estimation problem with given sample can be directly solved by selecting $b$ such that (denote the solution as $\hat{b}$)

$$
\frac{1}{T} \sum_{t=1}^{T} u_{t+1}(\hat{b})z_t = 0,
$$

where $T$ is the number of observations.

**Example. Sample mean and variance a GMM estimator:** Let $Y_1, \ldots, Y_T$ be a sample from a random variable with $E[Y] = \mu$, and $\text{Var}[Y] = \sigma^2$. Then $b = (\mu, \sigma^2)'$. Now $\sigma^2 = \text{Var}[Y] = E[(Y - \mu)^2] = E[Y^2] - \mu^2$. So that $E[Y^2] = \sigma^2 + \mu^2$. Then we can define

$$
\begin{pmatrix}
Y_t - \mu \\
Y_t^2 - (\sigma^2 + \mu^2)
\end{pmatrix}
$$

We have an exactly identified case, and using (40) with $z_t = 1$, we get

$$
\begin{pmatrix}
\frac{1}{T} \sum_{t=1}^{T} Y_t - \hat{\mu} \\
\frac{1}{T} \sum_{t=1}^{T} Y_t^2 - (\hat{\sigma}^2 + \hat{\mu}^2)
\end{pmatrix} = 0.
$$
Thus the GMM estimators of $\mu$ and $\sigma^2$ are

$$\hat{\mu} = \bar{Y} = \frac{1}{T} \sum_{t=1}^{T} Y_t,$$

the sample mean, and

$$\hat{\sigma}^2 = s^2 = \frac{1}{T} \sum_{t=1}^{T} Y_t^2 - \bar{Y}^2 = \frac{1}{T} \sum_{t=1}^{T} (Y_t - \bar{Y})^2,$$

the sample variance.

**Example. OLS estimator as a GMM estimator.**

$$y_t = \beta_0 + \beta_1 x_{1t} + \cdots + \beta_p x_{pt} + e_t$$

$$= x_t' b + e_t,$$

where $x_t = (1, x_{1t}, \ldots, x_{pt})'$ and $b = (\beta_0, \beta_1, \ldots, \beta_p)'$. Given observations, we can write

$$y = Xb + e,$$

where $y = (y_1, \ldots, y_T)'$, $X = (x_1, \ldots, x_T)'$: $T \times (p + 1)$ matrix of $x$-observations, and $e = (e_1, \ldots, e_T)'$. 

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In addition to
\[ E [y - Xb] = E [e] = 0, \]
an essential condition in regression is that \( E [X'e] = 0 \). That is the residuals are uncorrelated with the explanatory variables. Thus we can use \( X \) as instruments, so that (39) becomes
\[ E [(y_t - x'_t b)x_t] = 0. \]
Again we have an exactly identified case, and can write (40) as
\[ (y - X\hat{b})'X = 0, \]
from which we get
\[ (X'X)\hat{b} = X'y, \]
or
\[ \hat{b} = (X'X)^{-1}X'y, \]
the OLS estimator.
Finally if there are more equations (moments) in (39) than parameters in $b$ the estimation problem is over identified. The GMM estimator $b$ is then the one which satisfies (40) "as closely as possible". This is achieved by redefining (33) as

$$g_T(b) = \frac{1}{T} \sum_{t=1}^{T} u_t(b)z_t,$$

and applying (34).

Define a statistic, called the $J$-statistic as

$$J_T = g_T(\hat{b})'S^{-1}g_T(\hat{b}).$$

If the moment conditions are satisfied then asymptotically

$$TJ_T \sim \chi^2(df),$$

where $df = \#(\text{moments}) - \#(\text{parameters})$, i.e., number of overidentifying restrictions. This can be used to test the null hypothesis that the moment conditions (39) are satisfied, and is called a test for the overidentification restrictions.

Rejection of the null hypothesis indicates that the model does not fit the data.
Note. If in (39) $u_{t+1}(b)$ is a vector, then (39) must be expressed in a bit more general form

(44) \[ \mathbb{E} [u_{t+1}(b) \otimes z_t] = 0, \]

where $\otimes$ is the Kronecker product, which means that each element in $z_t$ are multiplied by $u_{t+1}(b)$. *

Example. Let $X_t = (X^a_t, X^b_t)' = (1 + R^a_t, 1 + R^b_t)'$ and instrument $z_t = (1, z_t)'$ then

\[
\mathbb{E} [(M_{t+1}(b)X_{t+1} - 1) \otimes z_t] = \mathbb{E} \begin{bmatrix}
M_{t+1}(b) X^a_{t+1} - 1 \\
M_{t+1}(b) X^b_{t+1} - 1 \\
M_{t+1}(b) X^a_{t+1} z_t - z_t \\
M_{t+1}(b) X^b_{t+1} z_t - z_t
\end{bmatrix} = 0.
\]

*If $a = (a_1, a_2)'$ and $c = (c_1, c_2)'$ then

\[
a \otimes c = \begin{pmatrix} a_1 c_1 \\ a_2 c_1 \\ a_1 c_2 \\ a_2 c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 c_1 \\ a_2 c_1 \\ a_1 c_2 \\ a_2 c_2 \end{pmatrix}
\]

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GMM in EViews

Application of GMM requires specification of the moment condition and instrument variables. Estimation can be applied both in single equation as well as system estimation. In EViews we simply define the moment conditions

\[ Mt+1(b)(1 + R_t) - 1 \]

and specify the instruments.

Example. Consider the consumption CAPM.