3. The Wiener Process and Rare Events in Financial Markets

The Wiener process works as a mathematical model for continuous time stock price.

To make the process even more realistic occasional rare events causing jumps into the sample paths are added by modeling with some point processes.

Consider first the modeling the "ordinary" events with Wiener process (Brownian motion) (see here for a heuristic construction of the Brownian motion).

**Definition 3.1**: A Wiener process, $W_t$, relative to a family of information sets $\{I_t\}$ (filtration), is a stochastic process such that
1. $W_0 = 0$ (with probability one).
2. $W_t$ is continuous.
3. $W_t$ is adapted to the filtration $\{I_t\}$.
4. For $s \leq t$, $W_t - W_s$ is independent of $I_s$, with $E[W_t - W_s] = 0$ and $\text{Var}[W_t - W_s] = E(W_t - W_s)^2 = t - s$.

Equivalently we can define:

**Definition 3.2**: A random process $B_t$, $t \in [0, T]$ is a (standard) Brownian motion if
1. $B_0 = 0$.
2. $B_t$ is continuous.
3. Increments of $B_t$ are independent. i.e., if
   $0 \leq t_0 < t_1 < \cdots < t_n$, then $B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.
4. If $0 \leq s \leq t$, $B_t - B_s \sim N(0, t - s)$.

Properties:
(a) $W_t$ is an $I_t$-martingale.
(b) $W_t$ has independent increments.
(c) $E[W_t] = 0$ for all $t$.
(d) $\text{Var}[W_t] = t$.
(e) The law of iterated logarithms
$$\limsup_{t \to \infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1 \quad \text{(with probability one)}$$
(f) $W_t/t \to 0$ w.p.1 as $t \to \infty$.
(g) $W_t^2 - t$ is an $I_t$-martingale.
(h) $\exp\{\sigma W_t - (\sigma^2/2)t\}$ is an $I_t$-martingale.

![Figure 3.1: Three sample paths of a standard Brownian motion.](image)
Proof: (a) Let $s \leq t$, then $\mathbb{E}[W_t - W_s | I_s] = \mathbb{E}[W_t - W_s] = 0$. So $\mathbb{E}[W_t | I_s] = W_s$.

(b) Let $0 \leq s < t < u$ then $W_t - W_s$ is $I_t$-measurable ($W_s, W_t \in I_t \subset I_u$) and $W_u - W_t$ is independent of $I_t$, which implies that $W_u - W_t$ is independent of $W_t - W_s$.

(c) $0 = \mathbb{E}[W_t - W_0] = \mathbb{E}[W_t]$.

(d) $\text{Var}[W_t] = \text{Var}[W_t - W_0] = t - 0 = t$.

(e) Will not be proven, but means that if $b > 1$, for sufficiently large $t$, $W_t < b\sqrt{2t \log \log t}$.

(f) Follows from the law of iterated logarithms.

(g) and (f) Left as exercises.

Example 3.1:

$$S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma W_t},$$

where $S_0 > 0$, and $\mu$ and $\sigma > 0$ are constants. This is an example of generalized Brownian motion, called also a Geometric Brownian motion, and serves as a basic model for stock prices. The distribution of $\log S_t$ is normal with mean

$$\log S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t,$$

and variance $\sigma^2 t$. The continuously compounded return to the stock, per unit of time (e.g. if annual, then annualized cumulative returns), over time interval $[t, t + u], u > 0$, is

$$R_u = \frac{1}{u} (\log S_{t+u} - \log S_t) = \mu - \frac{1}{2} \sigma^2 + \frac{1}{u} \sigma (W_{t+u} - W_t).$$

Because for fixed $t$, $W_{t+u} - W_t$ is a standard Brownian motion with time index $u$, property (f) above implies that

$$R_u \to \mu - \frac{1}{2} \sigma^2$$

as $u \to \infty$.

Note. $\mathbb{E}[S_t] = S_0 e^{\mu t}$. 

Figure 3.2: The law of iterated logarithms.

Figure 3.3: Sample paths of monthly stock price processes $S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma W_t}$, with $S_0 = 100$, and annual rate of return $\mu = 10\%$ and volatility $\sigma = 25\%$. 
Remark 3.1: In the above example the geometric Brownian motion is usually given in the differential form
\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,
\]
("model of the return") or
\[
dS_t = \mu S_t dt + \sigma S_t dW_t.
\]

## Rare Events and Poisson Process

In stock markets we occasionally observe sudden jumps in the prices due to some important or extreme event that affects the price. To model these a reasonable assumption could be that the jumps—being consequences of extreme shocks—are independent of the usual information driving innovations \(dW_t\).

Following Merton\(^*\) we can introduce a model where the asset price has jumps superimposed upon a geometric Brownian motion such that
\[
\frac{dS_t}{S_t} = (\mu - \lambda \nu) dt + \sigma dW_t + dJ_t,
\]
where \(\mu\) the asset’s expected return, \(\lambda\) is the jump intensity (e.g. average number of jumps per year), \(\nu\) is the average size of the jump as a percentage of the asset price, \(\sigma\) is the return volatility, and \(J_t\) is related to the (independent) Poisson process generating the jumps.

Once the jump occurs we can superimpose it into the process by assuming that it is generated from a normal distribution with mean \(\nu\) and standard deviation equaling the jump volatility.

Example 3.2: Consider a situation where the underlying process has a jump frequency one per year ($\lambda = 1/2$) (i.e., on average once every two year). The average percentage jump size $\nu$ is assumed for simplicity to be equal to zero and the volatility of the jump is 20%.

Suppose the stock price now is 100, the drift is 8%, $\sigma = 15\%$. Below are four sample paths from the diffusion process $dS_t = \mu dt + \sigma dW_t + dJ_t$.

We model $dJ_t$ as $dJ_t = 0.20 Z U$, where $Z$ is a standard normal variable and $U$ is a random variable such that $P(U = 1) = \lambda dt$ and $P(U = 0) = 1 - \lambda dt$. In addition $Z$ and $U$ are independent.

A discrete approximation of the above difference equation is

$$S_t = S_{t-h} \left( 1 + 0.08h + 0.15 \sqrt{h} Z + dJ_t \right).$$

We can rewrite the diffusion process in this example as

$$\frac{dS_t}{S_t} = (\mu - \lambda \nu) dt + \sigma dW_t$$

if the Poisson event does not occur, and

$$\frac{dS_t}{S_t} = (\mu - \lambda \nu) dt + \sigma dW_t + 0.20 Z$$

if the Poisson event occurs.

It can be shown that the process then is

$$S_t = S_0 \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 - \lambda \nu \right) t + \sigma W_t \right] J(N_t),$$

where

$$J(N_t) = \exp \left[ \sum_{i=1}^{N_t} J_i \right]$$

with $J_i = 0.20 Z_i$, and $Z_i$ are independent $N(0,1)$ random variables. $N_t$ has the Poisson distribution, discussed more closely below.

In order to model the jump intensity, let $N_t$ denote the total number of extreme (unordinary) shocks until time $t$ (counting process). Then once the event occurs, $N_t$ changes by one unit and remains otherwise unchanged.

Conceptually we can model this within an infinitesimal interval $dt$ by

(2) $dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases}$

This causes a discrete jump (once it occurs), because the size does not depend on $dt$ (the size can be a random variable as in the previous example).
Poisson process has the following properties:
(1) During a small interval $h$, at most one event occurs with probability $\approx 1$.
(2) The information up to time $t$ does not help to predict the occurrence of the event in the next instant.
(3) Events occur at a constant rate $\lambda$.

Remark 3.2: Within any time interval $[t, t+h]$, $\Delta N_{t+h} = N_{t+h} - N_t$ has the Poisson distribution

$P(\Delta N_{t+h} = k) = \frac{(\lambda h)^k}{k!} e^{-\lambda h}, \quad k = 0, 1, \ldots$

(4) $E[\Delta N_{t+h}] = \lambda h = \text{Var}[\Delta N_{t+h}]$.

However, if $h$ is small

$P(\Delta N_{t+h} = 1) = (\lambda h)e^{-\lambda h} \approx \lambda h$

as stated above. We denote $\Delta N_{t+h} \sim \text{Po}(\lambda h)$. Thus because $N_0 = 0$, we have in the previous example $N_t \sim \text{Po}(\lambda t)$.