7. Equivalent Martingale Measures

So far we have considered derivative asset pricing exploiting PDEs implied by arbitrage-free portfolios.

Another approach is to change the probability measure to another probability measure implied by arbitrage-free markets such that under that the (risk-free return discounted) prices become martingales.

As for background, consider pricing an European call option.

The aim is to find the fair price for the option given the available information.

To price the option \( (C_t) \), we use the best prediction of the end value in the light of available information, such that

\[
C_t = E_t [\rho \max(S_T - K, 0)],
\]

where \( E_t \) is the conditional expectation given information up to time \( t \), and \( \rho \) is a discount factor.
The no arbitrage theory implies that if the option is replicable, then the discount factor will be the riskfree rate, and the probability measure with respect to which the expectation must be calculated is such that the discounted price process 

\[ \tilde{S}_t = e^{-rt}S_t, \]

where \( r \) is the riskfree return, is martingale.

To illustrate the situation, consider the following single period discrete world.

Example 7.1: Suppose we have a call option \( C \) on stock \( S \) and a bank account \( B \). Let the exercise price of the option be \( K \), and assume that there are two possible end values \( S_1 > K > S_2 \) of the stock.

So

\[ \begin{align*}
S_0 & \rightarrow S_1 = uS_0, \text{ with } u > 1 \\
S_0 & \rightarrow S_2 = dS_0, \text{ with } d < 1,
\end{align*} \]

where \( p \) is the probability that the price goes up to \( S = uS_0 \), and \( S_0 \) is the current price of the stock.

Then the option with initial cost \( C \) has the end value \( \max\{S_1 - K, 0\} \). To replicate this with the stock and bank account with (riskfree) interest rate, \( r \), we may construct the following strategy:
Buy one share of the stock by financing it with cash and $S_2/(1+r)$ borrowed at rate $r$ from the bank (after one period the repayment is accordingly $S_2$). The value of the initial position is then

(3) 
$$S_0 - \frac{1}{1+r}S_2.$$ 

The end value of the position is according to the stock price as

<table>
<thead>
<tr>
<th></th>
<th>$S = S_1$</th>
<th>$S = S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock value</td>
<td>$S_1$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>Loan repayment</td>
<td>$-S_2$</td>
<td>$-S_2$</td>
</tr>
<tr>
<td>Total payoff</td>
<td>$S_1 - S_2$</td>
<td>0</td>
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</tbody>
</table>

We observe that in the case of $S = S_2$ the payoff is 0, the same as with the call options, and in the case $S = S_1$ the total payoff is $S_1 - S_2 = a(S_1 - K)$, where $a = (S_1 - S_2)/(S_1 - K)$. Thus in all, the payoff of the strategy is exactly the same as the payoff of a call options.

This implies that in the absence of arbitrage the cost of the investment must be the same in both cases. That is, buying a call options must have the same value as the other strategy based on one stock and bank loan.

So
$$aC = S_0 - \frac{1}{1+r}S_2$$

or
$$C = (S_0 - \frac{1}{1+r}S_2)/a = \frac{1}{1+r}p^*(S_1 - K),$$

where
$$p^* = \frac{1 + r - d}{u - d}.$$
We observe that $0 < p^* < 1$ (provided that $1 + r < u$), so that $p^*$ can be considered as a conditional probability given the initial price $S_0$ of the stock (the probability depends on $u$ and $d$ which are dependent on $S_0$, i.e., how much the price should go up to reach the given value $S_1$, or to decrease to go down to the other possible given value $S_2$). These probabilities are called risk neutral, hedging or martingale probabilities or probability measures. The last name is because they make the discounted price process $\tilde{S}$ a martingale.

This is seen as follows: We easily find that

(4) 
$$S_0 = \frac{1}{1+r} (p^* S_1 + (1 - p^*) S_2) = p^* \tilde{S}_1 + (1 - p^*) \tilde{S}_2,$$

where $\tilde{S} = S/(1 + r)$ is the discounted price process. That is, the conditional expectations of the discounted price process $\tilde{S}$ given the information $I_0 = \{S_0\}$ is

(5) 
$$E^*[\tilde{S}|I_0] = p^* \tilde{S}_1 + (1 - p^*) \tilde{S}_2 = \tilde{S}_0 = S_0,$$

so that $\tilde{S}$ is martingale with respect to the risk neutral probability measure, as stated above.
We observe that the option value does not at all depend on the true probabilities \( p \) and \( q = 1 - p \).

However, if we write the above martingale equation as 
\[
(q^* = 1 - p^*)
\]

\[
\mathbb{E}^*[\bar{S}|I_0] = p^*\bar{S}_1 + q^*\bar{S}_2 = \frac{p^*}{p}p\bar{S}_1 + \frac{q^*}{q}q\bar{S}_2,
\]

where \( p^*/p \) and \( q^*/q \) can be considered kinds of likelihood ratios or odds ratios, judged by the markets for the events that the stock price will be \( S_1 \) and \( S_2 \), respectively.

So the market expected value of the future stock price is a kind of likelihood weighted value of the possible future outcomes.

Translations of Probabilities

Probability Measure

As an illustration, consider the probability density \( f(z) \) of a standard normal distribution,
\[
f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.
\]

Then probability of that the random variable \( Z \) is near a specific value \( \bar{z} \) is
\[
P\left(\bar{z} - \frac{1}{2}\Delta < Z < \bar{z} + \frac{1}{2}\Delta\right) = \int_{\bar{z} - \frac{1}{2}\Delta}^{\bar{z} + \frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz,
\]

which is a real number (between zero and one).

Thus, the probability associates a real number (in this case between zero and one) to intervals on real line, or more generally to (Borel) sets.

Such functions are called measures in mathematics or measure functions.
Because $\Delta$ is small

$$
\int_{\bar{z} - \frac{1}{2}\Delta}^{\bar{z} + \frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \int_{\bar{z} - \frac{1}{2}\Delta}^{\bar{z} + \frac{1}{2}\Delta} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \Delta.
$$

(9)

For infinitesimal $\Delta$, denoted as $dz$, we designate the associated measure by symbol $dP(z)$, or simply by $dP$.

Thus, in the above case we have

$$
dP(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.
$$

(10)

Generally, if $P$ is a probability measure, we have

$$
\int_{-\infty}^{\infty} dP = 1.
$$

(11)

With these notations, e.g.,

$$
E[X] = \int_{-\infty}^{\infty} x dP(x).
$$

(12)

So the expected value is mathematically an integral with respect to probability measure. $dP$ is called sometimes the density of the probability measure $P$. 


Changing Probability Measure

Martingale model is a central tool for modeling fair prices of derivative securities.

However, generally, if $S_t$ is a risky asset, then given information up to time point $t$, we have

\[
E_t[S_{t+h}] > (1 + r_f)S_t, \quad (h > 0),
\]

because investors want some compensation for the risk, where $E_t$ is the conditional expectation, and $r_f$ is the risk-free rate.

However, we observed in the PDE approach that under the arbitrage-free pricing the risk-free rate should be a proper discounting factor in pricing risky derivative assets.

More importantly, the fundamental theorem of asset pricing establishes the equivalence of the absence of arbitrage opportunity and existence of martingale measure in (the stochastic model of) financial markets.
A probability measure, $\tilde{P}$, is a martingale measure for the discounted price process $\tilde{S}_t = e^{-rt}S_t$, if $\tilde{S}_t$ is martingale under $\tilde{P}$, i.e.

$$E^\tilde{P}_t[\tilde{S}_s] = \tilde{S}_t, \text{ whenever } s \geq t,$$

where $r$ is the riskfree rate, and $E^\tilde{P}_t$ indicates that the conditional expectation is taken with respect to probability measure $\tilde{P}$.

This theory implies that the no-arbitrage price of a contingent claim with underlying security $S_t$ and (random) payoff $X$ at maturity $T$ is obtained by

$$C_t = E^\tilde{P}_t[e^{-r\tau}X],$$

where $\tau = T - t$, and $\tilde{P}$ is the martingale measure for the discounted price process $\tilde{S}_t$.

We say that $\tilde{S}_t$ is $\tilde{P}$-martingale.

Thus a martingale measure can be viewed as a representation of the market's current opinion on the evolution of values of underlying assets and the prices of all derivatives contingent to them.

Consequently, the knowledge of the martingale measure is all that is needed, in principle, to value whatever derivative securities by the formula of the form (15).
Then given a stock price process $S_t$ with probability measure $P$, the goal is to find the martingale measure $\tilde{P}$.

This can be accomplished if there is an invertible function (one-to-one) $\xi(z)$ such that

$$d\tilde{P}(z) = \xi(z) dP(z).$$

(16)

Now recalling from standard calculus; if $G(z) = \int g(z) dz$ then $g(z) = dG(z)/dz = G'(z)$, i.e., $g(z)$ is the (mathematical) derivative of $G(z)$.

Alternatively, we can write $dG(z) = g(z) dz$, and hence

$$G(z) = \int dG(z) = \int g(z) dz.$$

(17)

In the same manner, because

$$\tilde{P}(z) = \int_{-\infty}^{z} d\tilde{P}(t) = \int_{-\infty}^{z} \xi(t) dP(t),$$

so that we can adopt notation

$$\frac{d\tilde{P}(z)}{dP(z)} = \xi(z),$$

(19)

and call $\xi$ as a derivative of $\tilde{P}$ with respect to $P$.

In mathematical measure theory this is known as the \textit{Radon-Nikodym derivative}. 


The existence of $\xi$ is guaranteed if $P$ and $\tilde{P}$ satisfy

\begin{equation}
\tilde{P}(dz) > 0 \text{ if and only if } P(dz) > 0.
\end{equation}

Actually this condition guarantees besides the existence of $\xi$ (known as Radon-Nikodym Theorem), also the equivalence of $\tilde{P}$ and $P$ in the sense

\begin{equation}
d\tilde{P}(z) = \xi(z) dP(z)
\end{equation}

and

\begin{equation}
dP(z) = \xi(z)^{-1}d\tilde{P}(z).
\end{equation}

In this sense $\tilde{P}$ and $P$ are equivalent probability measures.

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Example 7.2: (GBM) Consider the geometric Brownian motion

\begin{equation}
dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \quad t \geq 0
\end{equation}

so that

\begin{equation}
Z_t = \log \left( \frac{S_t}{S_0} \right) = \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t
\end{equation}

and

\begin{equation}
Z_t \sim N(\mu^* t, \sigma^2 t),
\end{equation}

where $\mu^* = \mu - \sigma^2/2$.

Thus

\begin{equation}
dP(z) = \frac{1}{\sqrt{2\pi} \sigma^2 t} e^{-\frac{(z - \mu^* t)^2}{2\sigma^2 t}} dz.
\end{equation}
Let
\[ Z_t = \left( r_f - \frac{1}{2} \sigma^2 \right) t + \sigma \tilde{W}_t, \] (27)
where \( r_f \) is a riskfree rate, and \( \tilde{W}_t \) is another Wiener process (the next example shows the relationship between \( W \) and \( \tilde{W} \)).

Then
\[ d\tilde{P}(\tilde{z}) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(\tilde{z} - r^*)^2}{2\sigma^2}}d\tilde{z}, \] (28)
where \( r^* = r_f - \frac{1}{2}\sigma^2 \).

Because the density function of the normal distribution is always positive, trivially
\[ P(dz) > 0 \iff \tilde{P}(d\tilde{z}) > 0. \] (29)

The transformation between these two probability measures is
\[ \xi(z) = e^{-\frac{(z - r^*)^2}{2\sigma^2}}, \] (30)
so that
\[ d\tilde{P}(z) = \xi(z) dP(z). \] (31)
The Girsanov Theorem

The Radon-Nikodym theorem gives the conditions under which the derivative $\xi$ exist (i.e., that we can move to another probability measure without losing any information, which means that those events that have positive probability have also positive probability under the other measure).

The \textit{Girsanov Theorem} provides the conditions under which the Radon-Nikodym derivative exists for cases where $Z_t$ is a continuous stochastic process.

For the purpose, let $\{I_t\}, t \in [0, T]$ be a family of information sets ($T < \infty$). Define

$$\xi_t = e^{\int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du}, \quad t \in [0, T],$$

where $X_t$ is an $I_t$-measurable process (that is once $I_t$ is given, the value of $X_t$ is known), and $W_t$ is a Wiener process with probability measure $P$. 
It is assumed that $X_t$ does not increase too fast, so that

\[(33) \quad E \left[ e^{\int_0^t X_u \, du} \right] < \infty, \quad t \in [0, T],\]

called Novikov condition (after a Russian mathematician).

Using Itô

\[(34) \quad d\xi_t = \xi_t X_t \, dW_t,\]

from which we immediately see that $\xi_t$ is a martingale, because $W_t$ is a Wiener process and there is no drift component in (34).

This is also easy to see formally.

Obviously, from (32)

\[(35) \quad \xi_0 = 1.\]

Thus,

\[(36) \quad \xi_t = \xi_0 + \int_0^t \xi_s X_s \, dW_s = 1 + \int_0^t \xi_s X_s \, dW_s.\]

\[(37) \quad E \left[ \int_0^t \xi_s X_s \, dW_s | \mathcal{I}_u \right] = \int_0^u \xi_s X_s \, dW_s, \quad u < t,\]

i.e., $\int_0^t \xi_s X_s \, dW_s$ is a martingale, and hence

\[(38) \quad E[\xi_t | \mathcal{I}_u] = 1 + \int_0^u \xi_s X_s \, dW_s = \xi_u,\]

implying that $\xi_t$ is a martingale.
Theorem. (Girsanov) Let $W_t$ be a Wiener process w.r.t probability measure $P$ and w.r.t information sets $\mathcal{I}_t$, and let $X_t$ be as defined above. Then if the process $\xi_t$ is a martingale w.r.t information sets $\mathcal{I}_t$, then $\tilde{W}_t$ defined by

$$\tilde{W}_t = W_t - \int_0^t X_u \, du, \quad t \in [0,T]$$

(39) is a Wiener process w.r.t information sets $\mathcal{I}_t$ and w.r.t probability measure

$$\tilde{P}(A) = \mathbb{E}^P[1_A \xi_T],$$

(40) where $A \in \mathcal{I}_T$ and $1_A$ is the indicator function of the event $A$.

In heuristic terms: If $W_t$ is a Wiener process with probability measure $P$, then

$$d\tilde{W}_t = dW_t - X_t \, dt$$

(41) is a Wiener process with probability measure $\tilde{P}$, such that $d\tilde{P} = \xi_T \, dP$.

Remark 7.1: Generally the theorem gives us a method to find the (equivalent) probability measures with respect to which a drifting process can be turned to a martingale.

Remark 7.2: We only change the drift and live the volatility intact.
Example 7.3: Consider the general diffusion process
\[ dS = a(S, t)dt + b(S, t)dW, \]
where \( W \) is the Wiener process w.r.t the probability measure \( P \),
\[ dP(w) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw, \]
the normal distribution \( N(0, t) \).

Define
\[ X_t = \frac{a(S, t)}{b(S, t)} \]
and assume that the drift \( a(S, t) \) and diffusion \( b(S, t) \)
are such that the Novikov condition (33) holds for \( X_t \).

Then defining
\[ \tilde{W} = W - \int_0^t \frac{a(S, u)}{b(S, u)} du \]
is a Wiener process with respect to the probability \( \tilde{P} \)
given by (40) and
\[ dS = b(S, t)d\tilde{W} \]
is a martingale w.r.t the probability measure \( \tilde{P} \).

Example 7.4: As in Example 7.2, consider again the geometric Brownian motion
\[ dS = \mu S dt + \sigma S dW, \]
where \( \mu \) and \( \sigma \) are constants. The probability measure for \( W \) is again (43).

Let \( r \) be the risk-free rate and consider the discounted price series
\[ \tilde{S}_t = e^{-rt} S_t. \]

Using Ito,
\[ d\tilde{S} = (\mu - r) \tilde{S} dt + \sigma \tilde{S} dW. \]
Now (44) becomes simply
\[ X_t = \frac{(\mu - r) \tilde{S}}{\sigma \tilde{S}} = \frac{\mu - r}{\sigma}, \]
\[ \tilde{W} = W - \frac{\mu - r}{\sigma} t \]
is Wiener process w.r.t the probability measure \( \tilde{P} \), and (w.r.t. this measure)
\[ d\tilde{S} = \sigma \tilde{S} d\tilde{W} \]
is martingale.
\[ \tilde{P} \]
In the Girsanov theorem
\begin{equation}
\xi_t = e^{\int_0^t X_u dW_u - \frac{1}{2} \int_0^t X_u^2 du}
= e^{\frac{1}{2}(\mu - r) \int_0^t (\mu - r)^2 du}.
\end{equation}

Furthermore, we can consider only the events $A = \{W_t \leq w\}$, $w \in \mathbb{R}$ (because here the information sets on the real line are Borel sets that are essentially open intervals).

Because $W_t \sim N(0, t)$ the associated probability measure is (43). Then
\begin{equation}
P(A) = \mathbb{E}^P[1_A \xi_t] = \int_{-\infty}^{w} \xi_t(u) \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du = \int_{-\infty}^{w} \xi(u) dP(u),
\end{equation}
i.e.,
\begin{equation}
d\check{P}(w) = \xi_t(w) dP(w) = e^{\frac{1}{2}(r-\mu) t - \frac{1}{2}(r-\mu)^2 t} dP(w)
= e^{\frac{1}{2}(r-\mu) w - \frac{1}{2}(r-\mu)^2 t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{w^2}{2t}} dw
= \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2}(r-\mu) w - \frac{1}{2}(r-\mu)^2 t} dw.
\end{equation}

Denoting
\begin{equation}
\tilde{w} = w - \frac{\mu - r}{\sigma} t,
\end{equation}
we have finally
\begin{equation}
d\check{P}(\tilde{w}) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\tilde{w}^2} d\tilde{w},
\end{equation}
which is again the density of the $N(0, t)$ distribution.

The end result is that discounted price process (48) is martingale with respect to the probability measure $\check{P}$.

The solution of (48) is
\begin{equation}
\check{S}_t = \check{S}_0 e^{-\frac{1}{2}\sigma^2 t + \sigma d\tilde{W}_t}.
\end{equation}

In terms of the original process from (48) $S_t = e^{rt} \check{S}_t$, we would have
\begin{equation}
S_t = S_0 e^{(r - \frac{1}{2}\sigma^2) t + \sigma d\tilde{W}_t},
\end{equation}
or
\begin{equation}
dS = rS dt + \sigma S d\tilde{W},
\end{equation}
i.e., we have essentially replaced the original drift $\mu$ with the risk free rate $r$, and the end result is a process whose discounted price process, $\check{S}_t = e^{-rt}S_t$ is a martingale. Process (58) is usually called the risk neutral process.
Remark 7.3: When pricing options, we calculate the expected values with respect to the distribution of risk neutral process $S_t$ given in (58). Its distribution is log-normal with density

$$f_{S_t}(y) = \frac{1}{\sqrt{2\pi t} \sigma y} e^{-\frac{(\log(y) - \tilde{\theta}_t)^2}{2\sigma^2 t}}, \quad y > 0,$$

where

$$\tilde{\theta}_t = \log S_0 + (r - \frac{1}{2}\sigma^2)t.$$ (61)

Remark 7.4: The original distribution of $S_t$ is log-normal with density

$$f_{S_t}(y) = \frac{1}{\sqrt{2\pi t} \sigma y} e^{-\frac{(\log(y) - \theta_t)^2}{2\sigma^2 t}}, \quad y > 0,$$

where

$$\theta_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t.$$ (63)