Financial Time Series Analysis: Part II

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1 Volatility Models

- Background
- Conditional Heteroskedasticity
  - ARCH-models
  - Properties of ARCH-processes
  - Estimation of ARCH models
  - Generalized ARCH models (GARCH)
  - ARCH-M Model
  - Asymmetric ARCH: TARCH, EGARCH, PARCH
  - The TARCH model
  - The EGARCH model
  - Power ARCH (PARCH)
  - Integrated GARCH (IGARCH)
- Predicting Volatility
- Realized volatility
Volatility Models

1. Background

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2. Predicting Volatility

- Realized volatility
Example 1

Consider the following daily close-to-close SP500 values.
For example a rolling $k = 22$ days ($\approx$ month of trading days) mean

$$m_t = \frac{1}{k} \sum_{u=t-k+1}^{t} r_u,$$  

(1)

and volatility (annualized standard deviation)

$$s_t = \sqrt{\frac{252}{k} \sum_{u=t-k+1}^{t} (r_u - m_t)^2},$$  

(2)

t = k, \ldots, T$$
give visual idea of volatility dynamics of the returns.
Volatility Models

Background

S&P 500 Daily Closing Prices and Volatility
[Jan 3 2000 to Feb 8 2017]
S&P 500 Daily Returns and Volatility
[Jan 3 2000 to Feb 8 2017]

Return
22 day rolling average

Absolute return
22 day rolling volatility

Date

Return (%)
−10 −5 0 5 10

Volatility (% p.a.)
0 25 50 75

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Financial Time Series Analysis: Part II
Because squared observations are the building blocks of the variance of the series, the results suggest that the variation (volatility) of the series is time dependent.

This leads to the so called ARCH-family of models.³

Note: Volatility not directly observable!!

Methods:

a) Implied volatility

b) Realized volatility

c) Econometric modeling (stochastic volatility, ARCH)

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Predicting Volatility

Realized volatility
Consider a time series $y_t$ with mean $\mu = \mathbb{E}[y_t]$ and denote by $u_t = y_t - \mu$ the deviation of $y_t$ from its mean, $t = 1, \ldots, T$.

The conditional distribution of $u_t$ given information up to time point $t - 1$ is denoted as

$$u_t | \mathcal{F}_{t-1} \sim D(0, \sigma_t^2), \quad (3)$$

where $\mathcal{F}_t$ is the information available at time $t$ (usually the past values of $u_t; u_1, \ldots, u_{t-1}$), $D$ is an appropriated (conditional) distribution (e.g., normal or $t$-distribution).

Then $u_t$ follows an ARCH($q$) process if its (conditional) variance $\sigma_t^2$ is of the form

$$\sigma_t^2 = \text{var}[u_t | \mathcal{F}_{t-1}] = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2. \quad (4)$$
Furthermore, it is assumed that $\omega > 0$, $\alpha_i \geq 0$ for all $i$ and $\alpha_1 + \cdots + \alpha_q < 1$.

For short it is denoted $u_t \sim \text{ARCH}(q)$.

This reminds essentially an AR($q$) process for the squared residuals, because defining $\nu_t = u_t^2 - \sigma_t^2$, we can write

$$u_t^2 = \omega + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 + \nu_t. \quad (5)$$

Nevertheless, the error term $\nu_t$ is time heteroscedastic, which implies that the conventional estimation procedure used in AR-estimation does not produce optimal results here.
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Predicting Volatility

Realized volatility

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Financial Time Series Analysis: Part II
Consider (for the sake of simplicity) ARCH(1) process

\[ \sigma^2_t = \omega + \alpha u^2_{t-1} \]  

(6)

with \( \omega > 0 \) and \( 0 \leq \alpha < 1 \) and \( u_t | u_{t-1} \sim N(0, \sigma^2_t) \).

(a) \( u_t \) is white noise:

(i) Constant mean (zero):

\[ \mathbb{E}[u_t] = \mathbb{E}[\mathbb{E}_{t-1}[u_t]] = \mathbb{E}[0] = 0. \]  

(7)

Note \( \mathbb{E}_{t-1}[u_t] = \mathbb{E}[u_t | \mathcal{F}_{t-1}] \), the conditional expectation given information up to time \( t - 1 \).

\[ \text{The law of iterated expectations: } \text{Consider time points } t_1 < t_2 \text{ such that } \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}, \text{ then for any } t > t_2 \]

\[ \mathbb{E}_{t_1} \left[ \mathbb{E}_{t_2}[u_t] \right] = \mathbb{E}\left[ \mathbb{E}[u_t | \mathcal{F}_{t_2}] | \mathcal{F}_{t_1} \right] = \mathbb{E}\left[ u_t | \mathcal{F}_{t_1} \right] = \mathbb{E}_{t_1}[u_t]. \]

(8)
(ii) Constant variance: Using again the law of iterated expectations, we get

\[ \text{var}[u_t] = \mathbb{E}[u_t^2] = \mathbb{E}[\mathbb{E}_{t-1}[u_t^2]] = \mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha u_{t-1}^2] = \omega + \alpha \mathbb{E}[u_{t-1}^2] \]

\[ = \omega \left( 1 + \alpha + \alpha^2 + \cdots + \alpha^n \right) + \alpha^{n+1} \mathbb{E}[u_{t-n-1}^2] \]

\[ \xrightarrow{n \to \infty} 0, \text{ as } n \to \infty \]

\[ = \omega \left( \lim_{n \to \infty} \sum_{i=0}^{n} \alpha^i \right) = \frac{\omega}{1-\alpha}. \]
(iii) Autocovariances: Exercise, show that autocovariances are zero, i.e., $E[u_t u_{t+k}] = 0$ for all $k \neq 0$. (*Hint*: use the law of iterated expectations.)

(b) The unconditional distribution of $u_t$ is symmetric, but nonnormal:

(i) Skewness: Exercise, show that $E[u_t^3] = 0$.

(ii) Kurtosis: Exercise, show that under the assumption $u_t | u_{t-1} \sim N(0, \sigma_t^2)$, and that $\alpha < \sqrt{1/3}$, the kurtosis

$$E[u_t^4] = 3 \frac{\omega^2}{(1-\alpha)^2} \cdot \frac{1-\alpha^2}{1-3\alpha^2}. \quad (10)$$

*Hint*: If $X \sim N(0, \sigma^2)$ then $E[(X - \mu)^4] = 3(\sigma^2)^2 = 3\sigma^4$. 

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Because $(1 - \alpha^2)/(1 - 3\alpha^2) > 1$ we have that

$$\mathbb{E}[u_t^4] > 3 \frac{\omega^2}{(1 - \alpha)^2} = 3 [\text{var}[u_t]]^2,$$

we find that the kurtosis of the unconditional distribution exceed that what it would be, if $u_t$ were normally distributed. Therefore the unconditional distribution of $u_t$ is nonnormal and has fatter tails than a normal distribution with variance equal to $\text{var}[u_t] = \omega/(1 - \alpha)$. 
(c) **Standardized variables:**

- Write
  \[ z_t = \frac{u_t}{\sqrt{\sigma_t^2}} \]  
  \( (12) \)
  then \( z_t \sim \text{NID}(0, 1) \), i.e., normally and independently distributed.
- Thus we can always write
  \[ u_t = z_t \sqrt{\sigma_t^2}, \]  
  \( (13) \)
  where \( z_t \) independent standard normal random variables (strict white noise).
- This gives us a useful device to check after fitting an ARCH model the adequacy of the specification: Check the autocorrelations of the squared standardized series (see, Econometrics I).
Volatility Models

Conditional Heteroskedasticity

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- Predicting Volatility

- Realized volatility
Instead of the normal distribution, more popular conditional distributions in ARCH-modeling are $t$-distribution and the generalized error distribution (ged), of which the normal distribution is a special case.

The unknown parameters are estimated usually by the method of the maximum likelihood (ML).
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In practice the ARCH needs fairly many lags.

Usually far less lags are needed by modifying the model to

\[ \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2, \]  

(14)

with \( \omega > 0, \alpha > 0, \beta \geq 0, \) and \( \alpha + \beta < 1. \)

The model is called the Generalized ARCH (GARCH) model.

Usually the above GARCH(1,1) is adequate in practice.

Econometric packages call \( \alpha \) (coefficient of \( u_{t-1}^2 \)) the ARCH parameter and \( \beta \) (coefficient of \( \sigma_{t-1}^2 \)) the GARCH parameter.
Applying backward substitution, one easily gets

\[
\sigma_t^2 = \frac{\omega}{1-\beta} + \alpha \sum_{j=1}^{\infty} \beta^{j-1} u_{t-j}^2
\]  

(15)

an ARCH(\infty) process.

Thus the GARCH term captures all the history from \( t - 2 \) backwards of the shocks \( u_t \).
Example 2

MA(1)-GARCH(1,1) model of SP500 returns estimated with conditional normal,
\( u_t | \mathcal{F}_{t-1} \sim N(0, \sigma_t^2) \), and GED, \( u_t | \mathcal{F}_{t-1} \sim GED(0, \sigma_t^2) \)

The model is

\[
\begin{align*}
    r_t &= \mu + u_t + \theta u_{t-1} \\
    \sigma_t^2 &= \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2.
\end{align*}
\]  

(16)
Volatility Models

Conditional Heteroskedasticity

Generalized ARCH models (GARCH)
Using R-program with package fGarch
(see http://cran.r-project.org/web/packages/fGarch/index.html)

garchFit(formula = ret ~ arma(2, 0) + garch(1, 1), data = sp1, trace = FALSE)

Conditional Distribution: norm
Coefficient(s):
\[
\begin{align*}
\text{mu} & : 0.052160 \\
\text{ar1} & : 0.059418 \\
\text{ar2} & : -0.028463 \\
\text{omega} & : 0.020165 \\
\text{alpha1} & : 0.100842 \\
\text{beta1} & : 0.883535
\end{align*}
\]

Std. Errors:
\[
\text{based on Hessian}
\]

Error Analysis:
\[
\begin{align*}
\text{Estimate} & \quad \text{Std. Error} \quad t \text{ value} \quad \text{Pr}(>|t|) \\
\text{mu} & : 0.052160 \quad 0.012800 \quad 4.075 \quad 4.6\times10^{-5} \text{ ***} \\
\text{ar1} & : -0.059418 \quad 0.016363 \quad -3.631 \quad 0.000282 \text{ ***} \\
\text{ar2} & : -0.028463 \quad 0.016126 \quad -1.765 \quad 0.077569 . \\
\text{omega} & : 0.020165 \quad 0.003255 \quad 6.196 \quad 5.8\times10^{-10} \text{ ***} \\
\text{alpha1} & : 0.100842 \quad 0.009422 \quad 10.703 \quad < 2\times10^{-16} \text{ ***} \\
\text{beta1} & : 0.883535 \quad 0.010146 \quad 87.079 \quad < 2\times10^{-16} \text{ ***}
\end{align*}
\]

---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Log Likelihood:
-6041.114 normalized: -1.404257
AIC: 2.811304 BIC: 2.820184
Volatility Models

Conditional Heteroskedasticity

Generalized ARCH models (GARCH)

garchFit(formula = ret ~ arma(2, 0) + garch(1, 1), data = sp1, cond.dist = "ged", trace = FALSE)

Conditional Distribution: ged

Coefficient(s):

| Coefficient | Estimate | Std. Error | t value | Pr(>|t|) |
|-------------|----------|------------|---------|----------|
| mu          | 0.071250 | 0.011961   | 5.957   | 2.57e-09 *** |
| ar1         | -0.060384| 0.014028   | -4.305  | 1.67e-05 *** |
| ar2         | -0.032524| 0.015550   | -2.092  | 0.0365 *    |
| omega       | 0.016393 | 0.003613   | 4.538   | 5.69e-06 *** |
| alpha1      | 0.101238 | 0.011276   | 8.978   | < 2e-16 *** |
| beta1       | 0.887963 | 0.011672   | 76.078  | < 2e-16 *** |
| shape       | 1.355355 | 0.041604   | 32.578  | < 2e-16 *** |

---

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Log Likelihood:
-5959.447 normalized: -1.385274
AIC: 2.773801 BIC: 2.784161
Goodness of fit (AIC, BIC) is marginally better for GED. The shape parameter estimate is $< 2$ (statistically significantly) indicated fat tails.

Otherwise the coefficient estimates are about the same.
Residual Diagnostics:

Autocorrelations of squared standardized residuals:

<table>
<thead>
<tr>
<th>Lag</th>
<th>Normal:</th>
<th>GED:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB  p-value</td>
<td>LB  p-value</td>
</tr>
<tr>
<td>1</td>
<td>4.306651 0.03796362</td>
<td>3.878418 0.04891062</td>
</tr>
<tr>
<td>2</td>
<td>4.823842 0.08964290</td>
<td>4.404404 0.11055943</td>
</tr>
<tr>
<td>3</td>
<td>4.843815 0.18359760</td>
<td>4.436248 0.21804779</td>
</tr>
<tr>
<td>5</td>
<td>6.009314 0.30531370</td>
<td>5.535524 0.35406622</td>
</tr>
<tr>
<td>10</td>
<td>16.268283 0.09220573</td>
<td>15.493970 0.11506346</td>
</tr>
</tbody>
</table>

JB = 232.8991, df = 2, p-value = 0.000
(dropping the outlier JB = 43.96)
Volatility Models

Conditional Heteroskedasticity

Generalized ARCH models (GARCH)

SP500 Return residuals [MA(1)-GED-GARCH(1,1)]

Conditional Volatility

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Autocorrelations of the squared standardized residuals pass (approximately) the white noise test.

Nevertheless, the normality of the standardized residuals is strongly rejected.

Usually this affects mostly to standard errors. Common practice is to use some sort of robust standard errors (e.g. White).
The variance function can be extended by including regressors (exogenous or predetermined variables), $x_t$, in it

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2 + \pi x_t.$$  \hfill (17)

Note that if $x_t$ can assume negative values, it may be desirable to introduce absolute values $|x_t|$ in place of $x_t$ in the conditional variance function.

For example, with daily data a Monday dummy could be introduced into the model to capture the non-trading over the weekends in the volatility.
Example 3

Monday effect in SP500 returns and/or volatility?

"Pulse" (additive) effect on mean and "innovative" effect on volatility

\[
y_t = \phi_0 + \phi_m M_t + \phi(y_{t-1} - \phi_m M_{t-1}) + u_t
\]

\[
\sigma_t^2 = \omega + \pi M_t + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2,
\]

where \( M_t = 1 \) if Monday, zero otherwise.
Generalized ARCH models (GARCH)

SAS: conditional t-distribution

```sas
proc autoreg data = tmp;
  model sp500 = mon/ nlag = 1 garch = (p=1, q=1) dist = t;
  hetero mon;
run;
```

The AUTOREG Procedure

GARCH Estimates

| Variable | DF | Estimate | Error | t Value | Pr > |t|   | Variable Label |
|----------|----|----------|-------|---------|-------|-----|----------------|
| Intercept| 1  | 0.0332   | 0.0187| 1.77    | 0.0765|     |                |
| mon      | 1  | 0.0110   | 0.0485| 0.23    | 0.8202|     |                |
| AR1      | 1  | 0.0669   | 0.0230| 2.91    | 0.0036|     |                |
| ARCH0    | 1  | 0.006221 | 0.002781| 2.24   | 0.0253|     |                |
| ARCH1    | 1  | 0.0742   | 0.0105| 7.04    | <.0001|     |                |
| GARCH1   | 1  | 0.9251   | 0.0102| 90.91   | <.0001|     |                |
| TDFI     | 1  | 0.1059   | 0.0172| 6.16    | <.0001|     | Inverse of t DF |
| HET1     | 1  | 6.452E-23| 1.002E-10| 0.00   | 1.0000|     |                |
Degrees of freedom estimate: $1/0.1059 \approx 9.4$.

No empirical evidence of a Monday effect in returns or volatility.
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The regression equation may be extended by introducing the variance function into the equation

\[ y_t = x'_t \theta + \lambda g(\sigma^2_t) + u_t, \quad (19) \]

where \( u_t \sim \text{GARCH} \), and \( g \) is a suitable function (usually square root or logarithm).

This is called the ARCH in Mean (ARCH-M) model [Engle, Lilien and Robbins (1987)]\(^5\).

The ARCH-M model is often used in finance, where the expected return on an asset is related to the expected asset risk.

The coefficient \( \lambda \) reflects the risk-return tradeoff.

Example 4

Does the daily mean return of SP500 depend on the volatility level?

Model AR(1)-GARCH(1, 1)-M with conditional $t$-distribution

$$y_t = \phi_0 + \phi_1 y_{t-1} + \lambda \sqrt{\sigma^2_t} + u_t$$

$$\sigma^2_t = \omega + \alpha u^2_{t-1} + \beta \sigma^2_{t-1},$$

where $u_t | \mathcal{F}_{t-1} \sim t_\nu$ ($t$-distribution with $\nu$ degrees of freedom, to be estimated).

```plaintext
proc autoreg data = tmp;
  model sp500 = / nlag = 1
    garch = (p=1, q=1, mean = sqrt)
    dist = t; /* t-dist */
run;
```
Volatility Models
Conditional Heteroskedasticity

ARCH-M Model

The AUTOREG Procedure

GARCH Estimates
SSE 4566.9544 Observations 2321
MSE 1.96767 Uncond Var .
Log Likelihood -2110.3437 Total R-Square 0.0058
SBC 4274.93573 AIC 4234.68746
MAE 0.93278599 AICC 4234.73588
MAPE 140.289693 Normality Test 330.0091
Pr > ChiSq <.0001

Standard Approx
Variable DF Estimate Error t Value Pr > |t|
Intercept 1 0.0429 0.0435 0.99 0.3245
AR1 1 0.0670 0.0230 2.91 0.0036
ARCH0 1 0.006174 0.002776 2.22 0.0262
ARCH1 1 0.0740 0.0105 7.03 <.0001
GARCH1 1 0.9253 0.0102 90.95 <.0001
DELTA 1 -0.009195 0.0483 -0.19 0.8489
TDFI 1 0.1064 0.0172 6.17 <.0001

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The volatility term in the mean equation (DELTA) is not statistically significant neither is the constant, indicating no discernible drift in SP500 index (again $\hat{\nu} = 1/0.1064 \approx 9.4$).
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A stylized fact in stock markets is that downward movements are followed by higher volatility than upward movements.

A rough view of this can be obtained from the cross-autocorrelations of $z_t$ and $z_t^2$, where $z_t$ defined in (12).
Volatility Models

Conditional Heteroskedasticity

Asymmetric ARCH: TARCH, EGARCH, PARCH

Example 5

Cross-autocorrelations of $z_t$ and $z_t^2$ from Ex 4

Some autocorrelations signify possible leverage effect.
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\[ \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \gamma u_{t-1}^2 d_{t-1} + \beta \sigma_{t-1}^2, \]  

(21)

where \( \omega > 0, \alpha, \beta \geq 0, \alpha + \frac{1}{2} \gamma + \beta < 1 \), and \( d_t = 1 \), if \( u_t < 0 \) (bad news) and zero otherwise.

The impact of good news is \( \alpha \) and bad news \( \alpha + \gamma \).

Thus, \( \gamma \neq 0 \) implies asymmetry.

Leverage exists if \( \gamma > 0 \).
Example 6

SP500 returns, MA(1)-TARCH model (EViews, www.eviews.com).
The TARCH model

Dependent Variable: SP500
Method: ML - ARCH (Marquardt) - Generalized error distribution (GED)
Included observations: 2321 after adjustments
Convergence achieved after 28 iterations
MA Backcast: 1/03/2000
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*RESID(-1)^2*(RESID(-1)<0) + C(6)*GARCH(-1)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Stat</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.012892</td>
<td>0.016635</td>
<td>0.774961</td>
<td>0.4384</td>
</tr>
<tr>
<td>MA(1)</td>
<td>-0.069378</td>
<td>0.022129</td>
<td>-3.135134</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

Variance Equation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Stat</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.010407</td>
<td>0.002084</td>
<td>4.993990</td>
<td>0.0000</td>
</tr>
<tr>
<td>RESID(-1)^2</td>
<td>-0.023073</td>
<td>0.008279</td>
<td>-2.786714</td>
<td>0.0053</td>
</tr>
<tr>
<td>RES(-1)^2*(RES(-1)&lt;0)</td>
<td>0.144410</td>
<td>0.016134</td>
<td>8.950672</td>
<td>0.0000</td>
</tr>
<tr>
<td>GARCH(-1)</td>
<td>0.939786</td>
<td>0.009180</td>
<td>102.3715</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

R-squared 0.007611 Mean dependent var -0.024928
Adjusted R-squared 0.007183 S.D. dependent var 1.407094
S.E. of regression 1.402032 Akaike info criterion 2.930426
Sum squared resid 4558.442 Schwarz criterion 2.947767
Log likelihood -3393.759 Hannan-Quinn criter. 2.936745
F-statistic 2.964170 Durbin-Watson stat 2.047115
Prob(F-statistic) 0.006937

Statistically significant positive asymmetry parameter estimate indicates presence of leverage.
1 Volatility Models

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- Predicting Volatility

- Realized volatility
Nelson (1991) (Econometrica, 347–370) proposed the Exponential GARCH (EGARCH) model for the variance function of the form (EGARCH(1,1))

\[
\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \alpha |z_{t-1}| + \gamma z_{t-1},
\]

(22)

where \( z_t = u_t / \sqrt{\sigma_t^2} \) is the standardized shock.

Again, the impact is asymmetric if \( \gamma \neq 0 \), and leverage is present if \( \gamma < 0 \).
Example 7

MA(1)-EGARCH(1,1)-M estimation results.
The EGARCH model

Dependent Variable: SP500
Method: ML - ARCH (Marquardt) - Generalized error distribution (GED)
Included observations: 2321 after adjustments
Convergence achieved after 25 iterations
MA Backcast: 1/03/2000
Presample variance: backcast (parameter = 0.7)

\[
\text{LOG(ARCH)} = C(3) + C(4)\times\frac{\text{ABS(RESID(-1)}/@\text{SQRT(GARCH(-1)))}} + C(5) \\
\quad \times \frac{\text{RESID(-1)}}{@\text{SQRT(GARCH(-1))}} + C(6)\times\text{LOG(GARCH(-1))}
\]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.017476</td>
<td>0.016027</td>
<td>1.090422</td>
<td>0.2755</td>
</tr>
<tr>
<td>MA(1)</td>
<td>-0.072900</td>
<td>0.021954</td>
<td>-3.320628</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

Variance Equation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(3)</td>
<td>-0.066913</td>
<td>0.013105</td>
<td>-5.106018</td>
<td>0.0000</td>
</tr>
<tr>
<td>C(4)</td>
<td>0.081930</td>
<td>0.016677</td>
<td>4.912873</td>
<td>0.0000</td>
</tr>
<tr>
<td>C(5)</td>
<td>-0.121991</td>
<td>0.012271</td>
<td>-9.941289</td>
<td>0.0000</td>
</tr>
<tr>
<td>C(6)</td>
<td>0.986926</td>
<td>0.002323</td>
<td>424.9402</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

GED PARAM 1.562564 0.051706 30.22032 0.0000

... 

Inverted MA Roots .07

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Volatility Models

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3. Predicting Volatility

4. Realized volatility

\[
\sigma_t^\delta = \omega + \beta \sigma_{t-1}^\delta + \alpha (|u_{t-1}| - \gamma u_{t-1})^\delta, \tag{23}
\]

where $\gamma$ is the leverage parameter. Again $\gamma > 0$ implies leverage.
Example 8

R: fGarch::garchFit MA(1)-APARCH(1,1) results for SP500 returns.

R parametrization: MA(1), mu and theta

```r
gfa <- fGarch::garchFit(sp500r~arma(0,1) + aparch(1,1), data = sp500r, cond.dist = "ged", trace=F)
```

Mean and Variance Equation:
```
data ~ arma(0, 1) + aparch(1, 1)
```

Conditional Distribution: ged

Error Analysis:

|       | Estimate | Std. Error | t value | Pr(>|t|) |
|-------|----------|------------|---------|----------|
| mu    | 0.014249 | 0.016825   | 0.847   | 0.397049 |
| ma1   | -0.076112| 0.021454   | -3.548  | 0.000389 *** |
| omega | 0.013396 | 0.003474   | 3.856   | 0.000115 *** |
| alpha1| 0.055441 | 0.008217   | 6.747   | 1.51e-11 *** |
| gamma1| 1.000000 | 0.016309   | 61.316  | < 2e-16 *** |
| beta1 | 0.935567 | 0.007620   | 122.775 | < 2e-16 *** |
| delta | 1.207412 | 0.203576   | 5.931   | 3.01e-09 *** |
| shape | 1.579328 | 0.068530   | 23.046  | < 2e-16 *** |

---

Signif. codes:  0 ***  0.001 **  0.01 *  0.05 .  0.1  1

Log Likelihood:  -3388.234 normalized:  -1.459816
The leverage parameter (‘gamma1’) estimates to unity. 
\( \hat{\delta} = 1.207412 \ (0.203576) \) does not deviate significantly from unity (standard deviation process).
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Often in GARCH $\hat{\alpha} + \hat{\beta} \approx 1$. Engle and Bollerslev (19896). Modelling the persistence of conditional variances, *Econometrics Reviews* 5, 1–50, introduce integrated GARCH with $\alpha + \beta = 1$.

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + (1 - \alpha)\sigma_{t-1}^2. \quad (24)$$

Close to the EWMA (Exponentially Weighted Moving Average) specification

$$\sigma_t^2 = \alpha u_{t-1}^2 + (1 - \alpha)\sigma_{t-1}^2 \quad (25)$$

favored often by practitioners (e.g. RiskMetrics).

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Predicting with the GARCH models is straightforward.

Generally a $k$-period forward prediction is of the form

$$\sigma^2_{t|k} = \mathbb{E}_t[u^2_{t+k}] = \mathbb{E}[u^2_{t+k}|\mathcal{F}_t]$$

$k = 1, 2, \ldots$
Because

\[ u_t = \sigma_t z_t, \]  \hspace{1cm} (27)

where generally \( z_t \sim \text{i.i.d}(0,1) \).

Thus in (26)

\[
\mathbb{E}_t[u^2_{t+k}] = \mathbb{E}_t[\sigma^2_{t+k} z^2_{t+k}]
\]

\[
= \mathbb{E}_t[\sigma^2_{t+k}] \mathbb{E}_t[z^2_{t+k}] \hspace{1cm} (z_t \text{ are i.i.d}(0,1)) \hspace{1cm} (28)
\]

\[
= \mathbb{E}_t[\sigma^2_{t+k}].
\]

This can be utilize to derive explicit prediction formulas in most cases.
ARCH(1):

\[ \sigma_{t+1}^2 = \omega + \alpha u_t^2 \]  

(29)

\[ \sigma_{t|k}^2 = \frac{\omega(1-\alpha^{k-1})}{1-\alpha} + \alpha^{k-1} \sigma_{t+1}^2 \] 

(30)

\[ = \sigma^2 + \alpha^{k-1}(\sigma_{t+1}^2 - \sigma^2), \]

where

\[ \sigma^2 = \text{var}[u_t] = \frac{\omega}{1-\alpha} \]  

(31)

Recursive formula:

\[ \sigma_{t|k}^2 = \begin{cases} \sigma_{t+1}^2 & \text{for } k = 1 \\ \omega + \alpha \sigma_{t|k-1}^2 & \text{for } k > 1 \end{cases} \]  

(32)
GARCH(1,1):

\[ \sigma_{t+1}^2 = \omega + \alpha u_t^2 + \beta \sigma_t^2. \] (33)

\[ \sigma_{t|k}^2 = \frac{\omega(1-(\alpha+\beta)^{k-1})}{1-(\alpha+\beta)} + (\alpha + \beta)^{k-1}\sigma_{t+1}^2 \] (34)

\[ = \sigma^2 + (\alpha + \beta)^{k-1}(\sigma_{t+1}^2 - \sigma^2), \]

where

\[ \sigma^2 = \frac{\omega}{1 - \alpha - \beta}. \] (35)

Recursive formula:

\[ \sigma_{t|k}^2 = \begin{cases} \sigma_{t+1}^2 & \text{for } k = 1 \\ \omega + (\alpha + \beta)\sigma_{t|k-1}^2 & \text{for } k > 1 \end{cases} \] (36)
IGARCH:

\[ \sigma_{t+1}^2 = \omega + \alpha u_t^2 + (1 - \alpha) \sigma_t^2. \]  

(37)

\[ \sigma_{t|k}^2 = (k - 1) \omega + \sigma_{t+1}^2. \]  

(38)

Recursive formula:

\[ \sigma_{t|k}^2 = \begin{cases} 
\sigma_{t+1}^2 & \text{for } k = 1 \\
\omega + \sigma_{t|k-1}^2 & \text{for } k > 1 
\end{cases} \]  

(39)
TGARCH:

\[ \sigma_{t+1}^2 = \omega + \alpha u_t^2 + \gamma u_t^2 d_t + \beta \sigma_t^2. \]  
(40)

\[ \sigma_{t|k}^2 = \frac{\omega(1-(\alpha+\frac{1}{2}\gamma+\beta)^{k-1})}{1-(\alpha+\frac{1}{2}\gamma+\beta)} + (\alpha + \frac{1}{2}\gamma + \beta)^{k-1}\sigma_{t+1}^2 \]  
(41)

\[ = \sigma^2 + (\alpha + \frac{1}{2}\gamma + \beta)^{k-1}(\sigma_{t+1}^2 - \sigma^2) \]

with

\[ \sigma^2 = \frac{\omega}{1 - \alpha - \frac{1}{2}\gamma - \beta}. \]  
(42)

Recursive formula:

\[ \sigma_{t|k}^2 = \begin{cases} 
\sigma_{t+1}^2 & \text{for } k = 1 \\
\omega + (\alpha + \frac{1}{2}\gamma + \beta)\sigma_{t|k-1}^2 & \text{for } k > 1 
\end{cases} \]  
(43)
EGARCH and APARCH prediction equations are a bit more involved.

Recursive formulas are more appropriate in these cases.

The volatility forecasts are applied for example in Value At Risk computations.
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Under certain assumptions, volatility during a period of time can be estimated more and more precisely as the frequency of the returns increases.

Daily (log) returns $r_t$ are sums of intraday returns (e.g. returns calculated at 30 minutes interval)

$$r_t = \sum_{h=1}^{m} r_t(h)$$  \hspace{1cm} (44)

where $r_t(h) = \log P_t(h) - \log P_t(h - 1)$ is the day’s $t$ intraday return in time interval $[h - 1, h]$, $h = 1, \ldots, m$, $P_t(h)$ is the price at time point $h$ within the day $t$, $P_t(0)$ is the opening price and $P_t(m)$ is the closing price.
The *realized variance* for day $t$ is defined as

$$\hat{\sigma}^2_{t,m} = \sum_{h=1}^{m} r_t^2(h)$$

and the *realized volatility* is $\hat{\sigma}_{t,m} = \sqrt{\hat{\sigma}^2_{t,m}}$ which is typically presented in percentages per annum (i.e., scaled by the square root of the number of trading days and presented in percentages).

Under certain conditions it can be shown that $\hat{\sigma}_{t,m} \to \sigma_t$ as $m \to \infty$, i.e., when the intraday return interval $\to 0$.


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Financial Time Series Analysis: Part II