A Short Introduction to the Generalized Method of Moments Estimation

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1. Introduction

Background

The most important mathematical results in probability theory are the Central Limit Theorem (CLT) and the (Weak) Law of Large Numbers (LLN). (Click here for a recap of stochastic convergence concepts). The CLT is related to the asymptotic distribution, and LLN to convergence of the estimates to the parameter value as the sample size increases.

The Central Limit Theorem (Classic version):

Let $y_1, y_2, \ldots$ be independent random variables with finite means, $\mathrm{E}[y_i] = \mu$, and finite variances, $\mathrm{Var}[y_i] = \sigma^2$. Then

$$\frac{\sqrt{n}(\bar{y}_n - \mu)}{\sigma} \xrightarrow{d} z \sim N(0, 1), \text{ as } n \to \infty$$

where

$$\bar{y}_n = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

(Weak) Law of Large Numbers

Let $y_1, y_2, \ldots$ be independent random variables with common mean $\mathrm{E}[y_i] = \mu$ and common variance $\mathrm{Var}[y_i] = \sigma^2$. Then

$$\operatorname{plim} \bar{y}_n = \mu,$$

where plim is the limit in probability as $n \to \infty$, denoted also as $\bar{y}_n \xrightarrow{p} \mu$, and

$$\bar{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

Properties of estimators

Let $y_1, y_2, \ldots, y_n$ be a sample from a distribution governed by parameter θ.

A function $t_n(\mathbf{y})$, where $\mathbf{y} = (y_1, \ldots, y_n)$, is a sample statistic if it is a function of sample observations alone. An estimator is a sample statistic whose purpose is to estimate an underlying parameter.

An estimator $t_n(\mathbf{y})$ is:

- unbiased if $\mathsf{E}[t_n(\mathbf{y})] = \theta$. If $\mathsf{E}[t_n(\mathbf{y})] \neq \theta$, $B\left(t_n(\mathbf{y})\right) = \mathsf{E}[t_n(\mathbf{y})] - \theta$ is called the bias of the estimator $t_n(\mathbf{y})$.

- minimum variance unbiased (MVUE) if it is unbiased and $\mathsf{Var}[t_n(\mathbf{y})] \leq \mathsf{Var}[t'_n(\mathbf{y})]$, where $t'_n(\mathbf{y})$ is any other unbiased estimator of θ.

Remark. If $t_n(\mathbf{y})$ is a vectors, then $\text{Var}[t_n(\mathbf{y})]$ is a covariance matrix and $\text{Var}[t_n(\mathbf{y})] \leq \text{Var}[t'_n(\mathbf{y})]$ means that the matrix $\text{Var}[t'_n(\mathbf{y})] - \text{Var}[t_n(\mathbf{y})]$ is positive semidefinite.

- consistent if $t_n(\mathbf{y}) \xrightarrow{p} \theta$.