## 2. Classical Method of Moments

Let $\theta$ be a $m$-vector of parameters that characterize the distribution random variable $y$. The $k^{th}$ moment (provided it exists) is defined as

$$\mu_k(\theta) = E\left[y^k\right].$$

Suppose we have sample of size $T$ from $y$ with observations $y_1, y_2, \ldots, y_T$ (considered as $T$ independent random variables). The corresponding sample moments are

$$\hat{\mu}_k = \frac{1}{T}\sum_{t=1}^{T} y_t^k.$$

With the method of moments one estimates the components of $\theta$ by simply equating the first $m$ population moments $\mu_k(\theta)$ with the corresponding sample moments $\hat{\mu}_k$, and solving for the components in the parameter vector $\theta$.

Example 2.1. Normal distribution, $y_1, \ldots, y_T$ a sample from $N(\mu, \sigma^2)$. Then $\theta = (\mu, \sigma^2)$,

$$\mu_1(\theta) = \mathsf{E}[y] = \mu,$$

and from $\sigma^2 = \mathsf{Var}[y] = \mathsf{E}[(y-\mu)^2] = \mathsf{E}[y^2] - \mu^2$,

$$\mu_2(\theta) = \mathsf{E}[y^2] = \sigma^2 + \mu^2.$$

Equating these with the sample moments, we have

$$\mu_1(\widehat{\theta}) = \frac{1}{T}\sum_{t=1}^{T} y_t$$

and

$$\mu_2(\widehat{\theta}) = \frac{1}{T}\sum_{t=1}^{T} y_t^2.$$

That is

$$\widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t = \bar{y}$$

and

$$\widehat{\sigma}^2 + \widehat{\mu}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2$$

Arrangin terms, we get for $\widehat{\sigma}^2$

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2 - \widehat{\mu}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2 - \bar{y}^2.$$

Recalling that $\sum_{t=1}^{T}(y_t - \bar{y})^2 = \sum_{t=1}^{T} y_t^2 - T\bar{y}$, we get finally

$$\hat{\mu} = \bar{y}$$

and

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})^2$$

as the MM estimators of $\mu$ and $\sigma^2$.

Example 2.2. The $t$-distribution. Suppose $y$ is a random variable following a $t$-distribution with $\nu$ degrees of freedom. The density is then

$$f(y_t;\nu) = \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi\nu}\Gamma(\nu/2)}[1 + (y_t/\nu)^2]^{-\frac{1}{2}(\nu+1)},$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

is the gamma function, with the property that, if $n$ is an integer ($\geq 0$) then $\Gamma(n+1) = n! = n\cdot(n-1)\cdots 2\cdot 1$, note that by definition $0! = 1$.

From sample $y_1,\ldots,y_T$, the MM estimator is obtained as follows. Provided that $\nu > 2$ the expected value of a $t$-distributed random variable is zero ($E[Y_t] = 0$) and variance

$$\mu_2 = E\left[(Y_t - E[Y_t])^2\right] = E\left[Y_t^2\right] = \frac{\nu}{\nu-2},$$

which at the same time is the second (population) moment of the distribution.

Equate this again with the second sample moment and solve for $\nu$ to obtain

$$\hat{\nu} = \frac{2\hat{\mu}_2}{\hat{\mu}_2 - 1}$$

provided that $\hat{\mu}_2 > 1$, where $\hat{\mu}_2 = (1/T)\sum_{t=1}^T y_t^2$. Otherwise the estimate does not exist.

The moment estimators are functions of averages of random variables. Thus the Law of Large Numbers implies that they converge toward their theoretical moments, implying consistency. Furthermore in many cases CLT implies that the asymptotic distributions of MM estimators are normal.