Monimuuttujaiset volatiliteettimallit
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Tavoite: Kurssilla perehdytään osakkeiden, valuuttakurssien, korkojen, jne. riskisyyden mittaamisessa käytetyn dynaamisen volatiiliteetin mallintamiseen useamman arvopaperin monimuuttujaisessa tapauksessa.

Luentoja ja harjoituksia 10 h

Sisältö:

1. Introduction to the modeling of security return volatility
2. Multivariate GARCH-models
3. Time dependent correlation models

1. Introduction to the modeling of security return volatility

1.1 Background

Below are time series plots of daily (log) returns

$$r_t = 100 \times (\log I_t - \log I_{t-1}),$$

where I_t is the index value at time point t, of HEX Forest and Metal industry indexes.

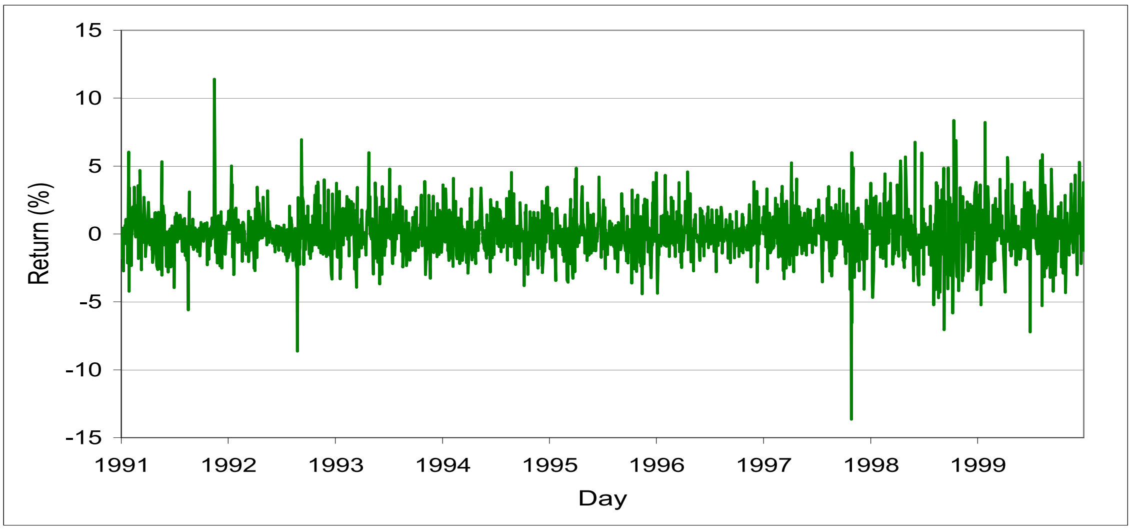


Figure 1.1. HEX forest industry daily index returns [January 2, 1990 to December 30, 1999].

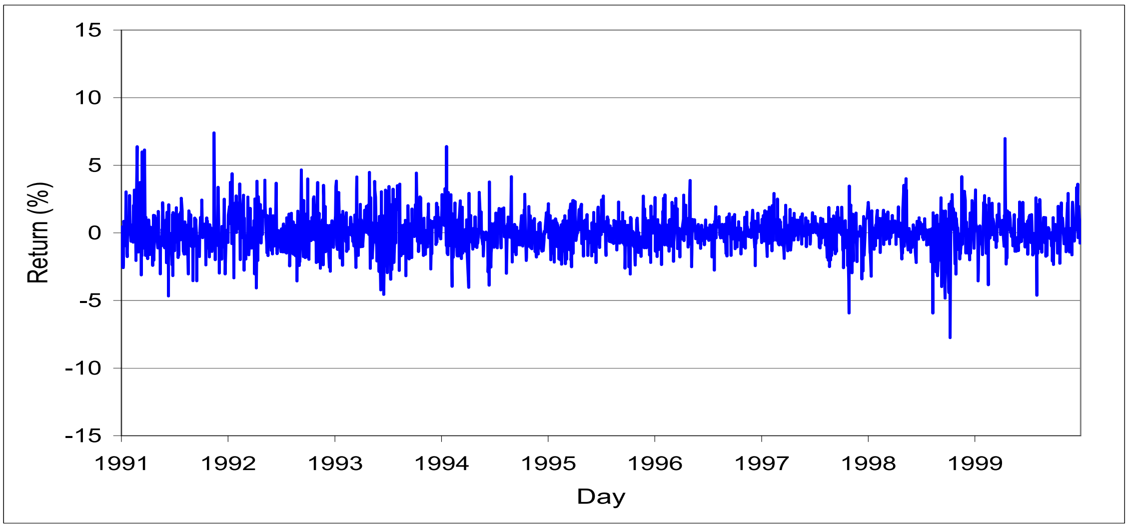


Figure 1.2. HEX metal industry daily index returns [January 2, 1990 to December 30, 1999].

We observe that level of volatility may change across time, and may be also be clustering. The volatility measured in terms of (annualized) 22-day rolling standard deviation

$$s_t = \sqrt{252 \times \frac{1}{22} \sum_{j=t-21}^{t} (r_j - \bar{r}_t)^2},$$

where

$$\bar{r}_t = \frac{1}{22} \sum_{j=t-21}^{t} r_j,$$

$t = \text{Jan } 31, 1990, \ldots, \text{Dec } 30, 1999$, are plotted in Figures 1.3 and 1.4.

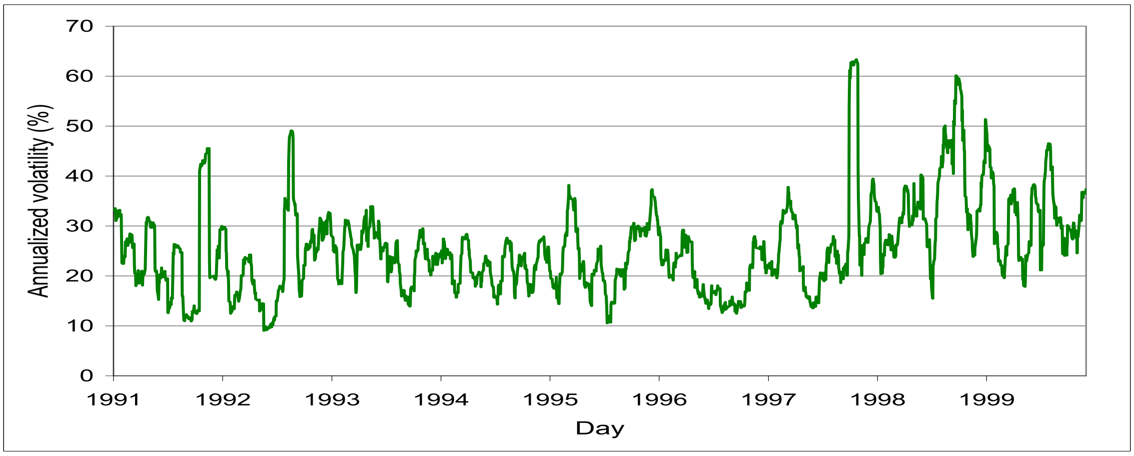


Figure 1.3. HEX forest industry rolling return volatility [January 31, 1990 to December 30, 1999].

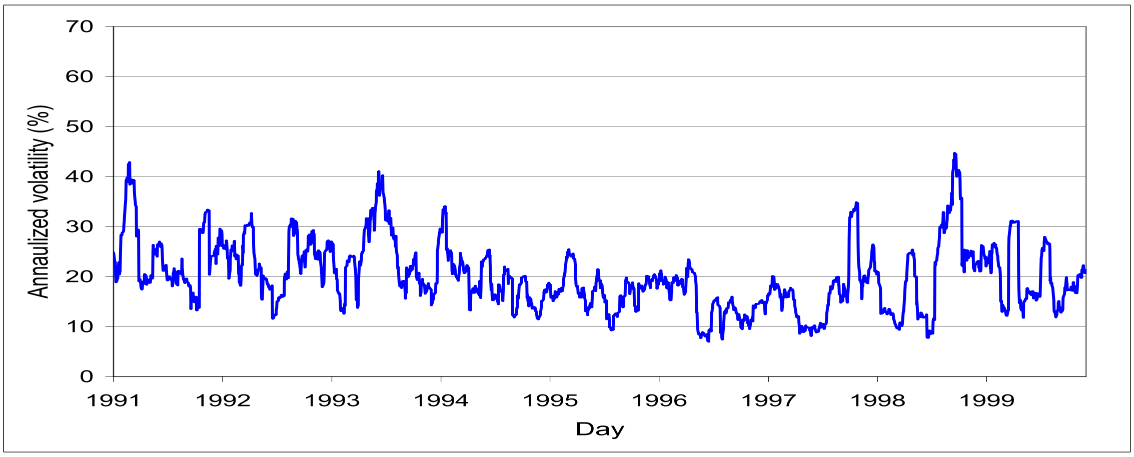


Figure 1.3. HEX metal industry rolling return volatility [January 31, 1990 to December 30, 1999].

Figure 1.4 depicts rolling 22 day correlation for the sample period.

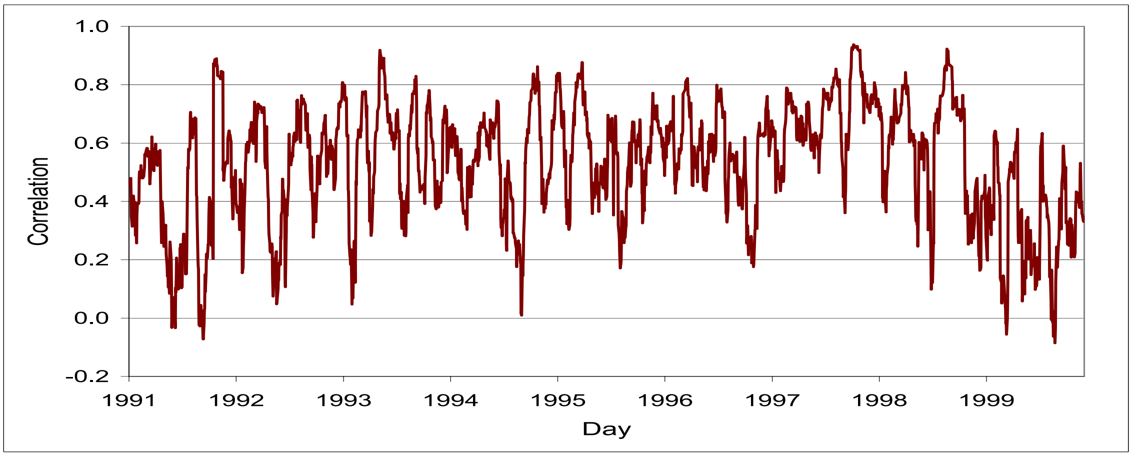


Figure 1.4. HEX forest and metal industry rolling contemporaneous return correlation [January 31, 1990 to December 30, 1999].

The above examples suggest that both volatility and contemporaneous correlations are time dependent. In next chapters we deal with some major approaches to capture the predictable patterns in these measures.

1.2 Conditional Heteroscedastic Models

Volatility is an important factor in *option pricing* and *risk management* (e.g. in Value at Risk, or VaR calculations). The difficulty is that the nature of volatility is not completely known.

In mathematical finance a popular continuous time model for the stock price S_t, $t \in [0, \infty)$, is the simple geometric Brownian motion (GBM)

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma, dW_t,$$

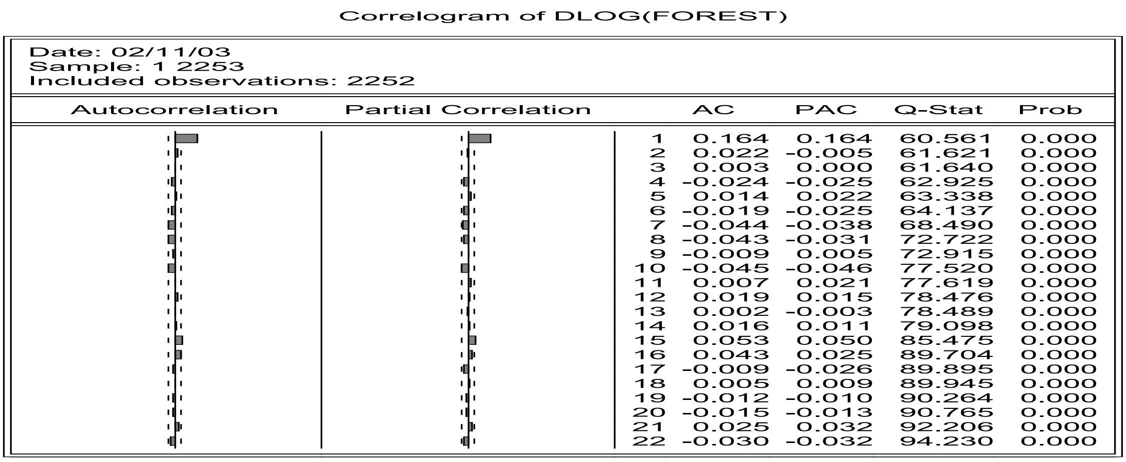
where μ is the drift (instantaneous return), σ is the diffusion (volatility) parameter, and W_t is the standard Brownian motion (or Wiener process).

Nevertheless, as *stylized facts*, it has been long observed that volatility:

- has clusters
- evolves over time in a continuous manner
- stays finite
- responses differently to price increases and price drops (asymmetry)

Structure of a stock return model

Generally stock returns have minor autocorrelation. Figure 1.5 presents autocorrelations of the HEX forest industry



Correlogram of DLOG(FOREST)

Date: 02/11/03
Sample: 1 2253
Included observations: 2252

	AC	PAC	Q-Stat	Prob
1	0.164	0.164	60.561	0.000
2	0.022	-0.005	61.621	0.000
3	0.003	0.000	61.640	0.000
4	-0.024	-0.025	62.925	0.000
5	0.014	0.022	63.338	0.000
6	-0.019	-0.025	64.137	0.000
7	-0.044	-0.038	68.490	0.000
8	-0.043	-0.031	72.722	0.000
9	-0.009	0.005	72.915	0.000
10	-0.045	-0.046	77.520	0.000
11	0.007	0.021	77.619	0.000
12	0.019	0.015	78.476	0.000
13	0.002	-0.003	78.489	0.000
14	0.016	0.011	79.098	0.000
15	0.053	0.050	85.475	0.000
16	0.043	0.025	89.704	0.000
17	-0.009	-0.026	89.895	0.000
18	0.005	0.009	89.945	0.000
19	-0.012	-0.010	90.264	0.000
20	-0.015	-0.013	90.765	0.000
21	0.025	0.032	92.206	0.000
22	-0.030	-0.032	94.230	0.000

Figure 1.5. HEX forest industry return autocorrelations with 95% confidence bands [January 3, 1990 to December 30, 1999].

There is slight first order autocorrelation ($\hat{\rho}_1 = 0.164$, well in this case moderate) in the returns.

denote available information at time point t, then we can model the return series as

$$r_t = \mu_t + u_t,$$

where

$$\mu_t = \mathsf{E}[r_t|\mathcal{F}_{t-1}],$$

$\mathcal{F}_{t-1}$ denotes the available information up to time point $t-1$, and u_t is the residual term.

Example. In the HEX forest case a useful model could be an AR(1). Estimation of the model yields

Dependent Variable: 100*DLOG(FOREST)
Method: Least Squares
Date: 02/10/03
Sample(adjusted): 3 2253
Included observations: 2251 after adjusting endpoints
Convergence achieved after 3 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.089127	0.042944	2.075403	0.0381
AR(1)	0.163922	0.020802	7.880002	0.0000

R-squared	0.026868	Mean dependent var	0.089149
Adjusted R-squared	0.026435	S.D. dependent var	1.726471
S.E. of regression	1.703498	Akaike info criterion	3.904133
Sum squared resid	6526.386	Schwarz criterion	3.909214
Log likelihood	-4392.102	F-statistic	62.09442
Durbin-Watson stat	1.997771	Prob(F-statistic)	0.000000

Inverted AR Roots .16

Residual autocorrelations reveal no evidence of further serial correlations

Correlogram of Residuals

Date: 02/10/03
Sample: 3 2253
Included observations: 2251
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.001	0.001	0.0015	
		2	-0.006	-0.006	0.0733	0.787
		3	0.004	0.004	0.1027	0.950
		4	-0.028	-0.028	1.9268	0.588
		5	0.021	0.021	2.9275	0.570
		6	-0.015	-0.015	3.4143	0.636
		7	-0.036	-0.035	6.2961	0.391
		8	-0.037	-0.038	9.3435	0.229
		9	0.005	0.006	9.4053	0.309
		10	-0.047	-0.049	14.499	0.106

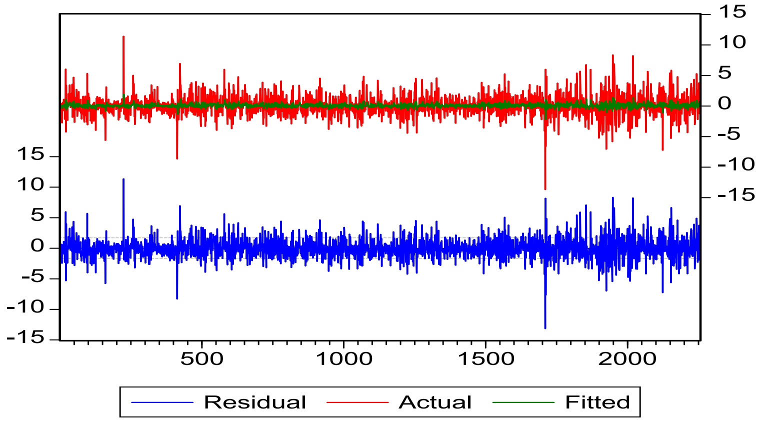
Nevertheless, the squared residuals are autocorrelated.

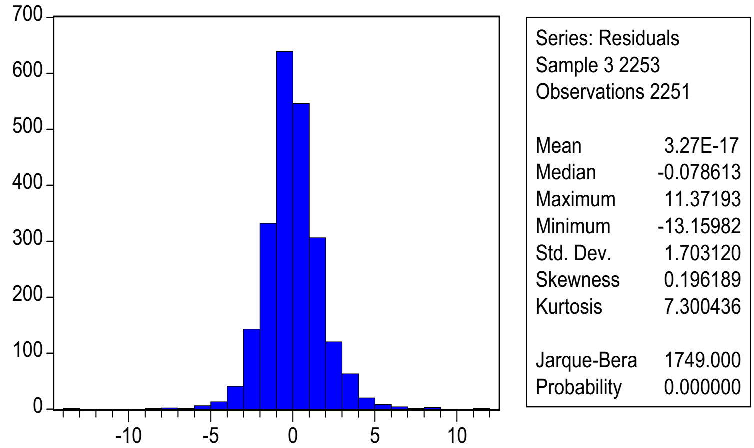
Correlogram of Standardized Residuals Squared

Date: 02/11/03
Sample: 3 2253
Included observations: 2251
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.007	-0.007	0.1058	
		2	0.068	0.068	10.399	0.001
		3	0.074	0.075	22.624	0.000
		4	0.039	0.036	26.084	0.000
		5	0.064	0.055	35.235	0.000
		6	-0.006	-0.015	35.306	0.000
		7	0.049	0.036	40.739	0.000
		8	0.042	0.035	44.713	0.000
		9	-0.001	-0.008	44.717	0.000
		10	0.090	0.077	62.878	0.000
		11	0.013	0.009	63.276	0.000
		12	0.037	0.021	66.309	0.000
		13	0.022	0.008	67.455	0.000
		14	0.017	0.007	68.134	0.000
		15	0.029	0.011	70.025	0.000
		16	0.028	0.024	71.864	0.000
		17	0.013	0.000	72.252	0.000
		18	0.028	0.015	74.050	0.000
		19	0.028	0.021	75.870	0.000
		20	0.004	-0.012	75.898	0.000
		21	0.026	0.015	77.459	0.000
		22	-0.011	-0.022	77.758	0.000

The residual time series and histogram look as follows





The typical phenomenon is that there are relatively many 'outliers'.

The squared residual indicate volatility. The next step is to model the volatility as a time dependent process such that

$$\begin{aligned}\sigma_t^2 &= \mathrm{var}[u_t|\mathcal{F}_{t-1}] \\ &= \mathrm{var}\left[r_t|\mathcal{F}_{t-1}\right] \\ &= \mathrm{E}\left[(r_t-\mu_t)^2|\mathcal{F}_{t-1}\right].\end{aligned}$$

ARCH Models

Engle (1982)* gave the first systematic framework for volatility modeling. The basic set up is that the mean corrected asset return $u_t = r_t - \mu_t$ is serially uncorrelated, but not serially independent. The basic model [ARCH(m)–AutoRegressive Conditional Heteroscedastic] is of the form

$$u_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t^2), \tag{1}$$

where $u_t|\mathcal{F}_{t-1}$ denotes the conditional random variable, and

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \cdots + \alpha_m u_{t-m}^2 \tag{2}$$

with $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \alpha_i < 1$.

*Engle, Robert, F. (1983). Autoregressive conditional heteroscedasticity with estimates of the variance of the United Kingdom inflations. *Econometrica*, **50**, 987–1007.

Using the conditional representation of the joint distribution, and considering the first usable observation as given, the log-likelihood function is

$$\ell(\theta) = \sum_{t=m+p+1}^{T} \ell_t(\theta), \tag{3}$$

where θ is the parameter vector including the p AR and m ARCH parameters, and

$$\ell_t(\theta) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma_t^2) - \frac{u_t^2}{2\sigma_t^2}, \tag{4}$$

where $u_t = r_t - \mu_t$. The parameters can be estimated numerically with the ML-method.

Note. $z_t = u_t/\sigma_t \sim \text{NID}(0, 1)$, i.e., the standardized residuals are normally and independently distributed. So we can always write

$$u_t = \sigma_t z_t, \tag{5}$$

where $z_t \sim \text{NID}(0, 1)$.

This also gives us a diagnostics tool to check the adequacy of the fitted ARCH-specification.

Remark. Often in literature h_t is used to denote the conditional *variance*, such that $h_t = \text{var}[u_t|\mathcal{F}_t]$. Then $u_t = \sqrt{h_t}\, z_t$.

Example. Consider the forest index return series. Estimation results for an AR(1)-ARCH(1) model are

Dependent Variable: 100*DLOG(FOREST)
Method: ML - ARCH (Marquardt)
Date: 02/11/03 Time: 13:19
Sample(adjusted): 3 2253
Included observations: 2251 after adjusting endpoints
Convergence achieved after 11 iterations
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.091533	0.044224	2.069732	0.0385
AR(1)	0.189907	0.023161	8.199373	0.0000
	Variance Equation			
C	2.389747	0.052271	45.71868	0.0000
ARCH(1)	0.177051	0.017189	10.30023	0.0000

R-squared	0.026192	Mean dependent var	0.089149
Adjusted R-squared	0.024891	S.D. dependent var	1.726471
S.E. of regression	1.704848	Akaike info criterion	3.867424
Sum squared resid	6530.923	Schwarz criterion	3.877587
Log likelihood	-4348.786	F-statistic	20.14510
Durbin-Watson stat	2.049117	Prob(F-statistic)	0.000000
Inverted AR Roots	.19		

Autocorrelations of the squared standardized residuals indicate yet serial dependence. Thus further modeling is needed.

Correlogram of Standardized Residuals Squared

Date: 02/11/03
Sample: 3 2253
Included observations: 2251
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.007	-0.007	0.1058	
		2	0.068	0.068	10.399	0.001
		3	0.074	0.075	22.624	0.000
		4	0.039	0.036	26.084	0.000
		5	0.064	0.055	35.235	0.000
		6	-0.006	-0.015	35.306	0.000
		7	0.049	0.036	40.739	0.000
		8	0.042	0.035	44.713	0.000
		9	-0.001	-0.008	44.717	0.000
		10	0.090	0.077	62.878	0.000
		11	0.013	0.009	63.276	0.000
		12	0.037	0.021	66.309	0.000
		13	0.022	0.008	67.455	0.000
		14	0.017	0.007	68.134	0.000
		15	0.029	0.011	70.025	0.000
		16	0.028	0.024	71.864	0.000
		17	0.013	0.000	72.252	0.000
		18	0.028	0.015	74.050	0.000
		19	0.028	0.021	75.870	0.000
		20	0.004	-0.012	75.898	0.000
		21	0.026	0.015	77.459	0.000
		22	-0.011	-0.022	77.758	0.000

One possibility is to increase the order of the ARCH, but in practice it has proved more useful to generalize the basic ARCH.

GARCH models

A natural extension of the basic GARCH suggested by Bollerslev (1986)* is

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2. \qquad (6)$$

We denote $u_t \sim \text{GARCH}(p, q)$.

*Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**, 307–327.

It is assumed that $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$.

In practice GARCH$(1,1)$ has proved to be many times adequate.

Example. Below are AR(1)-GARCH(1,1) estimation results for the forest index returns.

Dependent Variable: 100*DLOG(FOREST)
Method: ML - ARCH (Marquardt)
Date: 02/11/03
Sample(adjusted): 3 2253
Included observations: 2251 after adjusting endpoints
Convergence achieved after 23 iterations
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.094226	0.041785	2.255007	0.0241
AR(1)	0.175995	0.022053	7.980509	0.0000
Variance Equation				
C	0.382737	0.042530	8.999181	0.0000
ARCH(1)	0.114030	0.010271	11.10236	0.0000
GARCH(1)	0.755168	0.023044	32.77083	0.0000

R-squared	0.026716	Mean dependent var	0.089149
Adjusted R-squared	0.024983	S.D. dependent var	1.726471
S.E. of regression	1.704768	Akaike info criterion	3.832832
Sum squared resid	6527.404	Schwarz criterion	3.845535
Log likelihood	-4308.852	F-statistic	15.41298
Durbin-Watson stat	2.021686	Prob(F-statistic)	0.000000
Inverted AR Roots	.18		

Correlogram of Standardized Residuals Squared

Date: 02/11/03 Time: 15:08
Sample: 3 2253
Included observations: 2251
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.007	0.007	0.1258	
		2	-0.014	-0.014	0.5824	0.445
		3	0.012	0.012	0.8940	0.640
		4	-0.017	-0.018	1.5765	0.665
		5	-0.007	-0.007	1.7026	0.790
		6	-0.026	-0.027	3.2855	0.656
		7	-0.001	-0.001	3.2904	0.772
		8	-0.005	-0.006	3.3389	0.852
		9	-0.020	-0.019	4.2133	0.837
		10	0.038	0.038	7.5448	0.581
		11	-0.008	-0.009	7.6879	0.659
		12	0.002	0.002	7.6938	0.740
		13	-0.004	-0.006	7.7221	0.806
		14	0.006	0.007	7.7946	0.857
		15	-0.004	-0.005	7.8229	0.898
		16	0.006	0.009	7.9151	0.927
		17	0.000	-0.001	7.9152	0.951
		18	0.010	0.010	8.1206	0.964
		19	0.016	0.017	8.6858	0.967
		20	-0.016	-0.018	9.3019	0.968
		21	0.009	0.010	9.4721	0.977
		22	-0.016	-0.016	10.023	0.979

Autocorrelation in the squared standardized residuals disappears. In this sense the fit is OK.

Nevertheless, the standardized residuals cannot be inferred to be normal as indicated below by the Jarque-Bera statistic

$$JB = \frac{T-k}{6}\left(S^2 + \frac{(K-3)^2}{4}\right), \tag{7}$$

where k is the number of estimated parameters in the regression-GARCH model, S and K are the sample skewness and kurtosis of the standardized residuals. Under the null hypothesis of normality the JB statistic is asymptotically χ^2-distributed with 2 degrees of freedom. Particularly the kurtosis remains high.

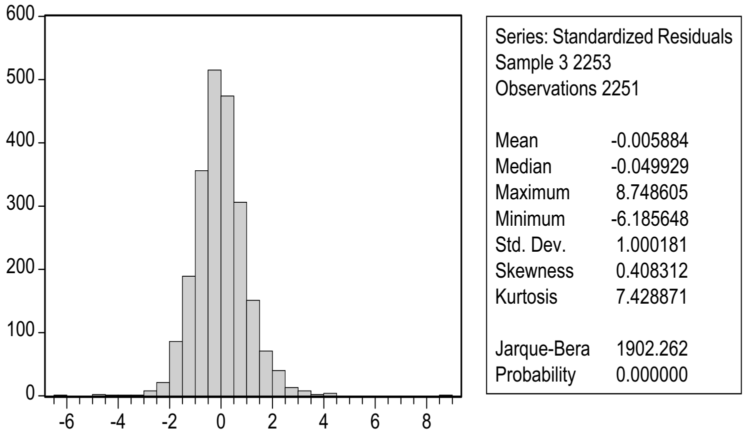


Figure. Histogram and sample statistics of the standardized AR(1)-GARCH(1,1) residuals.

To capture the excess kurtosis one can use some fatter tail distribution in the conditional model (1). One possibility is the t-distribution. If $x_\nu \sim t_\nu$, $\nu > 2$, then $\text{var}[x_\nu] = \nu/(\nu-2)$, and the standardized random variable $z_t = x_\nu/\sqrt{\nu/(\nu-2)}$ has density

$$f(z_t;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\left(1+\frac{z_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}} \tag{8}$$

given that $\nu > 2$. The degrees of freedom, ν, is usually an additional parameter to be estimated.

Using $u_t = \sigma_t z_t$ (and the change of variable technique), $\ell_t(\theta)$ in the log-likelihood representation (4) becomes

$$\begin{aligned}\ell_t(\theta) \;=\; & \log\Gamma\left(\tfrac{1}{2}(\nu+1)\right) \\ & -\tfrac{1}{2}\log\left((\nu-2)\pi\right) - \log\Gamma\left(\tfrac{1}{2}\nu\right) \\ & -\tfrac{1}{2}(\nu+1)\log\left(1+\tfrac{u_t^2}{(\nu-2)\sigma_t^2}\right) \\ & -\tfrac{1}{2}\log\sigma_t^2 .\end{aligned} \tag{9}$$

Another popular model is the so called Generalized Error Distribution (GED), but we do not consider it here.

Other extensions

Huge number of other extension of the basic ARCH models exists. Below are mentioned few most common ones

GARCH-M, Garch in mean

GARCH(1,1)-M

$$\begin{aligned} r_t &= \mu_t + \gamma g(\sigma_t^2) + u_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{aligned} \tag{10}$$

where $g(\cdot)$ is some function, usually the log or the square root.

Example. GARCH(1,1)-M for the forest industry returns

Dependent Variable: 100*DLOG(FOREST)
Method: ML - ARCH (Marquardt)
Date: 02/11/03
Sample(adjusted): 3 2253
Included observations: 2251 after adjusting endpoints
Convergence achieved after 24 iterations
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.217617	0.151409	1.437284	0.1506
C	-0.247963	0.245190	-1.011313	0.3119
AR(1)	0.171094	0.022211	7.702964	0.0000
		Variance Equation		
C	0.383000	0.043783	8.747632	0.0000
ARCH(1)	0.116940	0.010475	11.16366	0.0000
GARCH(1)	0.752564	0.023609	31.87641	0.0000

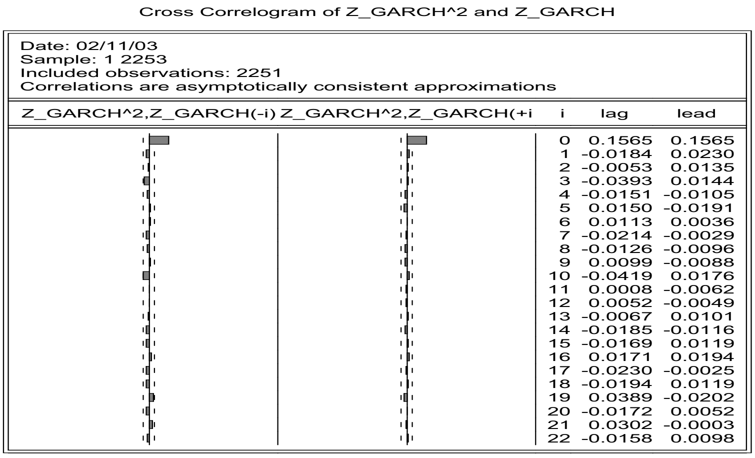
R-squared	0.026852	Mean dependent var	0.089149
Adjusted R-squared	0.024685	S.D. dependent var	1.726471
S.E. of regression	1.705029	Akaike info criterion	3.832969
Sum squared resid	6526.491	Schwarz criterion	3.848213
Log likelihood	-4308.006	F-statistic	12.38943
Durbin-Watson stat	2.014031	Prob(F-statistic)	0.000000
Inverted AR Roots	.17		

The results indicate that the mean component is not statistically significant. So the basic garch model is adequate so far.

Exponential GARCH, EGARCH

GARCH-model do not take into account possible asymmetry in volatility shocks. A quick way to check asymmetry in volatility is to check cross correlations of z_t^2 and z_t. If these indicate statistical significance there the asymmetry should be accounted for.

Example. Check the AR-GARCH residuals for asymmetry


Cross Correlogram of Z_GARCH^2 and Z_GARCH

Date: 02/11/03
Sample: 1 2253
Included observations: 2251
Correlations are asymptotically consistent approximations

Z_GARCH^2,Z_GARCH(-i)	Z_GARCH^2,Z_GARCH(+i	i	lag	lead
		0	0.1565	0.1565
		1	-0.0184	0.0230
		2	-0.0053	0.0135
		3	-0.0393	0.0144
		4	-0.0151	-0.0105
		5	0.0150	-0.0191
		6	0.0113	0.0036
		7	-0.0214	-0.0029
		8	-0.0126	-0.0096
		9	0.0099	-0.0088
		10	-0.0419	0.0176
		11	0.0008	-0.0062
		12	0.0052	-0.0049
		13	-0.0067	0.0101
		14	-0.0185	-0.0116
		15	-0.0169	0.0119
		16	0.0171	0.0194
		17	-0.0230	-0.0025
		18	-0.0194	0.0119
		19	0.0389	-0.0202
		20	-0.0172	0.0052
		21	0.0302	-0.0003
		22	-0.0158	0.0098

These preliminary checks indicate no evident existence of cross correlation between the squared standized and the standardized residuals.

EGARCH models aim to take into account the possible asymmetry (leverage). EGARCH(1,1) is defined as

$$\begin{aligned} \log(\sigma_t^2) &= \omega + \beta \log(\sigma_{t-1}^2) + \gamma z_{t-1} \\ &\quad + \delta \left(|z_{t-1}| - \mathrm{E}[|z_{t-1}|] \right). \end{aligned} \tag{11}$$

where $z_t = u_t/\sigma_t$ is i.i.d standardized random variable. If $z_t \sim N(0,1)$, then $\mathrm{E}[|z_t|] = \sqrt{2/\pi}$. In the case of t-distribution

$$\mathrm{E}[|z_t|] = \frac{2\sqrt{\nu-2}\,\Gamma\left(\frac{1}{2}(\nu+1)\right)}{(\nu-1)\Gamma\left(\frac{1}{2}\nu\right)\sqrt{\pi}}.$$

There are many other (equivalent) ways to specify the EGARCH model.

Negative sign of γ indicates that drop in stock price increases future volatility, and positive sign that increase in stock price causes additional volatility in the return series.

Example. EGARCH(1,1) specification for the forest index return series.

Dependent Variable: 100*DLOG(FOREST)
Method: ML - ARCH (Marquardt)
Date: 02/11/03
Sample(adjusted): 3 2253
Included observations: 2251 after adjusting endpoints
Convergence achieved after 34 iterations
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.071806	0.042779	1.678545	0.0932
AR(1)	0.185057	0.020005	9.250586	0.0000
Variance Equation				
C	-0.040524	0.007474	-5.421792	0.0000
|RES|/SQR[GARCH](1)	0.172904	0.016177	10.68824	0.0000
RES/SQR[GARCH](1)	-0.045739	0.009731	-4.700347	0.0000
EGARCH(1)	0.912027	0.012620	72.27099	0.0000

R-squared	0.026354	Mean dependent var	0.089149
Adjusted R-squared	0.024186	S.D. dependent var	1.726471
S.E. of regression	1.705465	Akaike info criterion	3.834749
Sum squared resid	6529.830	Schwarz criterion	3.849994
Log likelihood	-4310.010	F-statistic	12.15346
Durbin-Watson stat	2.039441	Prob(F-statistic)	0.000000
Inverted AR Roots	.19		

The estimate $\hat{\gamma} = -0.046$ is highly statistically significant, but small on abslolute value. So there is evidence of the leverage effect, though it may not have any practially discernible impact on volatility.

It may be noted that both AIC and BIC are slightly smaller in the case of the GARCH specification, which indicate that GARCH should be the preferable specification.

There are several other models to capture the leverage effect. Neverthelss we stop here and move to the multivariate case.