3. Dynamic Conditional Correlation

In the Dynamic Conditional Correlation (DCC) proposed by Engle (2002)* the conditional correlation is parameterized directly.

Let r_i, $i = 1, \ldots, n$ be random variables with zero mean. The conditional correlations are defined as

$$\rho_{ij,t} = \frac{\mathsf{E}_{t-1}[r_{i,t}\, r_{j,t}]}{\sqrt{\mathsf{E}_{t-1}[r_{i,t}^2]\mathsf{E}_{t-1}[r_{j,t}^2]}}. \tag{43}$$

Let $\sigma_{i,t}^2 = \mathsf{E}_{t-1}[r_{i,t}^2]$, then $z_{i,t} = r_{i,t}/\sigma_{it} \sim \mathsf{WN}(0,1)$, and the correlation can be written as

$$\rho_{ij,t} = \mathsf{E}_{t-1}[z_{i,t}\, z_{j,t}]. \tag{44}$$

Note: Although z_{it} are individually i.i.d it does not follw that they are jointly independend! That is why the conditional correlation may depend on the past. However, joint independnece implies independence of individual $z_{i,t}$s.

*Engle, Robert, F. (2002). Dynamic conditional correlation—A simple class of multivariate GARCH. *Journal of Business and Economics Statistics.* **17**, No. 5, 425–446.

Engle (2002) suggest to estimate the following GARCH process

$$q_{ij,t} = \bar{\rho}_{ij} + \alpha\,(z_{i,t-1}\, z_{j,t-1} - \bar{\rho}_{ij}) + \beta\,(q_{ij,t-1} - \bar{\rho}_{ij}), \tag{45}$$

and

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}\, q_{jj,t}}}, \tag{46}$$

where $\bar{\rho}_{ij}$ is the unconditional expectation of the cross product $z_{i,t}z_{j,t}$, i.e., unconditional correlations.

Thus the main difference here compared to the above approaches is that the correlations are modeled individually as GARCH processes *with common GARCH parameters α and β and separate unconditional expectations $\bar{\rho}_{ij}$ of the cross products.*

Thus there are altogether $n(n+1)/2 + n + 2$ parameters to be estimated in the model.

Using these and the separately modeled GARCH variance processes the resulting covariance matrix is obtained as

$$\mathbf{\Sigma}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t, \tag{47}$$

where D_t is the diagonal matrix of conditional standard deviations [c.f. formula (27)], and R_t is the conditional correlation matrix with elements given by (46).

<u>Remark</u>. The correlations in (46) will give a p.d. correlation matrix, because $Q_t = (q_{ij,t})$ is a weighted average of p.s.d and p.d matrix.

<u>Remark</u>. ω_{ij} is the unconditional correlation, and each term in the denominator of (46) has expected value one.

<u>Remark</u>. Model (45) is mean reverting as long as $\alpha + \beta < 1$ $(\alpha, \beta \geq 0)$

<u>Estimation</u>

A formulation of the DCC model is

$$\begin{aligned}
&\mathbf{r}_t|\mathcal{F}_{t-1} \sim N(0, \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t)\\
&\mathbf{D}_t = \mathsf{diag}(\sigma_{1,t}, \ldots, \sigma_{n,t})\\
&\mathbf{z}_t = \mathbf{D}_t^{-1}\mathbf{r}_t\\
&\mathbf{Q}_t = (q_{ij,t})\\
&\mathbf{R}_t = (\mathsf{diag}(\mathbf{Q}_t))^{-\frac{1}{2}}\, \mathbf{Q}_t\, (\mathsf{diag}(\mathbf{Q}_t))^{-\frac{1}{2}},
\end{aligned} \tag{48}$$

where $\sigma_{i,t} = \sqrt{\alpha_{0,i} + \alpha_{i,1} r_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2}$, $i = 1, \ldots, n$ are the GARCH(1,1) standard deviations, and
$(\mathsf{diag}(\mathbf{Q}_t))^{-\frac{1}{2}} = \mathsf{diag}(1/\sqrt{q_{11,t}}, \ldots, 1/\sqrt{q_{nn,t}})$.

Let θ denote the parameters in $\mathbf{D}_t$, and ϕ the parameters in $\mathbf{R}_t$, then the log likelihood function is

$$\ell(\theta, \phi) = \sum_{t=1}^{T} \ell_t(\theta, \phi), \tag{49}$$

where (ignoring the constant term $n \log 2\pi$)

$$\ell_t(\theta, \phi) = -\frac{1}{2}\left(\log|\mathbf{D}_t\mathbf{R}_t\mathbf{D}_t| + \mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{R}_t^{-1}\mathbf{D}_t^{-1}\mathbf{r}_t\right). \tag{50}$$

In order to simplify the maximization procedure it is accomplished in two steps. For the purpose, rearrange terms in (50) such that

$$\begin{aligned}
\ell_t(\theta, \phi) \;=\; &-\tfrac{1}{2}\big(2\log|\mathbf{D}_t| + \mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{D}_t^{-1}\mathbf{r}_t\\
&-\mathbf{z}_t'\mathbf{z}_t + \log|\mathbf{R}_t| + \mathbf{z}_t'\mathbf{R}_t^{-1}\mathbf{z}_t\big)
\end{aligned} \tag{51}$$

Thus we can decompose the likelihood function into volatility part

$$\ell_{V,t}(\theta) = -\frac{1}{2}\left(2\log|\mathbf{D}_t| + \mathbf{r}_t'\mathbf{D}_t^{-1}\mathbf{D}_t^{-1}\mathbf{r}_t\right), \tag{52}$$

and correlation part

$$\ell_{C,t}(\theta,\phi) = -\frac{1}{2}\left(\log|\mathbf{R}_t| + \mathbf{z}_t'\mathbf{R}_t^{-1}\mathbf{z}_t - \mathbf{z}_t'\mathbf{z}_t\right), \tag{53}$$

so that

$$\ell_t(\theta,\phi) = \ell_{V,t}(\theta) + \ell_{C,t}(\theta,\phi). \tag{54}$$

Engle (2002) suggest a two step procedure, where in the first step the variance part

$$\ell_V(\theta) = \sum_{t=1}^{T} \ell_{V,t}(\theta) \tag{55}$$

is maximized, and then given the maximizing value $\widehat{\theta}$, the correlation part

$$\ell_C(\widehat{\theta},\phi) = \sum_{t=1}^{T} \ell_{C,t}(\widehat{\theta},\phi) \tag{56}$$

is maximized with respect to ϕ.

Remark. Because $\mathbf{D}_t$ is a diagonal matrix, the variance part (55) is the sum of individual GARCH likelihoods

$$\ell_V(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\left(\log(2\pi) + \log(\sigma_{i,t}^2) + \frac{r_{i,t}^2}{\sigma_{i,t}^2}\right), \tag{57}$$

which implies that in the first step the GARCH models can be estimated separately to each series.

Remark. Because $\mathbf{z}_t'\mathbf{z}_t$ terms remain unchanged in (53) you can ignore it and simplify

$$\ell_{C,t}(\theta,\phi) = -\frac{1}{2}\left(\log|\mathbf{R}_t| + \mathbf{z}_t'\mathbf{R}_t^{-1}\mathbf{z}_t\right), \tag{58}$$

Example. Consider the Nordic stock indices. As noted above in the first step we simply estimate separate GARCH(1,1) models for each index return series. Because there is some autocorrelation in the return series, we adopt the following specifications

$$\begin{array}{rcl} y_{i,t} &=& \phi_{i,0} + \phi_{i,1}y_{i,t-1} + u_{i,t} \\ h_{i,t} &=& \omega_i + \alpha_i u_{i,t-1}^2 + \beta_i h_{i,t-1}, \end{array} \tag{59}$$

where in this case $h_{i,t} = \mathrm{var}_{t-1}[u_{i,t}]$ denotes the conditional variance, $i = 1,\ldots,4$ with $1 =$ Denmark, $2 =$ Finland, $3 =$ Norway, and $4 =$ Sweden.

RATS estimates for the AR and GARCH parameters are as follows

Variable	Coeff	Std Error	T-Stat	Signif
PHI0(1)	0.0258196495	0.0207576877	1.24386	0.21355123
PHI0(2)	0.0463076920	0.0366149726	1.26472	0.20597168
PHI0(3)	0.0572372304	0.0264378197	2.16498	0.03038957
PHI0(4)	0.0844326535	0.0209986432	4.02086	0.00005799
PHI1(1)	0.1652603392	0.0218878890	7.55031	0.00000000
PHI1(2)	0.1994514707	0.0194654520	10.24643	0.00000000
PHI1(3)	0.1843649422	0.0276705032	6.66287	0.00000000
PHI1(4)	0.1462929873	0.0169525559	8.62955	0.00000000
OMEGA(1)	0.0812695314	0.0360803362	2.25246	0.02429319
OMEGA(2)	0.0715641017	0.0280201692	2.55402	0.01064868
OMEGA(3)	0.0513847071	0.0247171604	2.07891	0.03762579
OMEGA(4)	0.0574232062	0.0176653233	3.25062	0.00115155
ALPHA(1)	0.1358098187	0.0352697921	3.85060	0.00011783
ALPHA(2)	0.1330450304	0.0231289321	5.75232	0.00000001
ALPHA(3)	0.1640905715	0.0434603942	3.77563	0.00015960
ALPHA(4)	0.1366265789	0.0234533241	5.82547	0.00000001
BETA(1)	0.7759450051	0.0611495501	12.68930	0.00000000
BETA(2)	0.8309721748	0.0316890857	26.22266	0.00000000
BETA(3)	0.8054756174	0.0505598297	15.93114	0.00000000
BETA(4)	0.8244948090	0.0251068472	32.83944	0.00000000

Thus e.g., for Finland

$$\begin{array}{rll} y_{2,t} = & \underset{(0.037)}{0.0463} + & \underset{(0.019)}{0.1995}y_{2,t-1} + \hat{u}_{2,t} \end{array}$$

$$h_{2,t} = \underset{(0.028)}{0.0716} + \underset{(0.023)}{0.1330}u_{2,t-1} + \underset{(0.032)}{0.8310}h_{2,t-1}$$

with standard errors in parentheses.

For the sake of simplicity we use the simplified GARCH(1,1) specification for the correlation part as

$$q_{ij,t} = \bar{\rho}_{i,j} + \alpha(z_{i,t-1}z_{j,t-1} - \bar{\rho}_{i,j}) + \beta(q_{ij,t-1} - \bar{\rho}_{ij}), \tag{60}$$

where $\bar{\rho}_{ij}$ are estimated by the sample contemporaneous correlations. That is, only the GARCH(1,1)-parameters α and β are estimated, and $\bar{\rho}_{ij}$ are replaced by the sample contemporaneus correlations of the standardized series $z_{i,t}$. The correlations are as follows

Contemporaneous standardized residual correlations

	Den	Fin	Nor	Swe
Den	1			
Fin	0.371	1		
Nor	0.461	0.416	1	
Swe	0.464	0.504	0.542	1

These are generally a little bit lower than the return contemporaneous correlations considered earlier.

The (quasi) ML esitmation of the GARCH(1,1) parameters yields $\hat{\alpha} = 0.0336(0.001703)$ and $\hat{\beta} = 0.9247(0.004116)$ with standard errors in parentheses. The respective t-values are 19.7 and 224.7, and thus highly significant.

The estimate of β is close to one, which implies that the correlations should be highly persistent. Nevertheless, as indicated by the graphs below there is considerable variation in the correlations. The general tendency is increasing. These can be compared with the earlier unconditional contemporaneous correlations

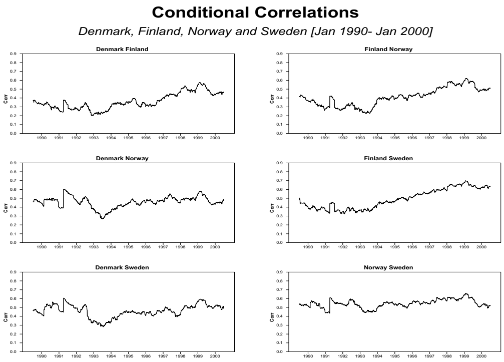

Figure. Estimated dynamic conditional correlations.

Bellow is RATS code. If experimented with another data sets only those parts indicated by $' <<<<'$ should be changed.¶

```
*
*  Estimated Dynamic Conditional Correlation DCC Example
*
Cal(Irregular)
*
COMPUTE GSTART=4, GEND = 2547 ;*   <<<<
all gend
Open Data ind012000pg.xls ;*       <<<<
data(format=xls,org=cols)
comp nSeries = 4 ;* number of seies in the analysis <<<<
* Define return series
dec vect[series] y(nSeries) ;* Return series
set y(1) = 100*log(deClose/deClose{1}) ;*       <<<<
set y(2) = 100*log(fiClose/fiClose{1}) ;*       <<<<
set y(3) = 100*log(noClose/noClose{1}) ;*       <<<<
set y(4) = 100*log(swClose/swClose{1}) ;*       <<<<
*
*  YOU NEED NOT CHANGE THE PROGRAM BELOW THIS
*
*  Parameters for the regression function
*
dec vect phi0(nSeries) phi1(nSeries) ;* AR-parameters
dec vect omega(nSeries) alpha(nSeries) beta(nSeries) ;* GARCH params
dec vect yt(nSeries)
dec vect yt1(nSeries)
dec vect[series] z(nSeries) ;* standardized series
dec vect[series] h(nSeries) ;* GARCH processes
dec vect ht(nSeries)
dec vect ht1(nSeries)
dec frml[vect] resid ;* AR residual vector functions
dec vect residt1(nSeries)
dec frml[vect] hf ;* variance vector funcitons
dec symm sigma(nSeries,nSeries)
dec vec zVec(nSeries)
```

¶I want to thank Martin Richter from Danskebank for careful examination of an earlier version of the code and helpful suggestions.


```
NONLIN(parmset=arParms) phi0 phi1
NONLIN(parmset=garchParms) omega alpha beta
FRML RESID = yt=%xt(y,t),yt1=%xt(y,t-1),yt - phi0 - %diag(phi1)*yt1
frml hf    = residt1=%xt(z,t-1),ht1=%xt(h,t-1),$
             residt1=%xdiag(%outerxx(residt1)),$
             omega + %diag(alpha)*residt1 + %diag(beta)*ht1
frml glogl = %pt(z,t,resid(t)),$
             %pt(h,t,hf(t)),$
             sigma = %diag(%xt(h,t)), zVec=%xt(z,t),$
             %logdensity(sigma,zVec)
*
*  Do initial AR regression.
*  Copy initial values for regression parameters, and
*  Initialize GARCH variance series
do i=1,nSeries
 LINREG(noPrint) y(i) / z(i)
 # CONSTANT y(i){1}
 COMPUTE phi0(i) = %BETA(1), phi1(i) = %BETA(2)
 set h(i) = %seesq**2
end do i
*
* Initialize GARCH estimates
comp omega = alpha = beta = %mscalar(0.05)
*
* Estimate GARCH parameters
MAXIMIZE(parmset=arParms+garchparms,$
METHOD=SIMPLEX,ITERS=10) GLOGL GSTART GEND ;* Improve starting val
MAXIMIZE(parmset=arParms+garchparms,$
METHOD=BFGS,ITERS=100,robust) GLOGL GSTART GEND
```



```
* CORRELATION PART
*
* Standardized variables
*
do i=1,nSeries
 set z(i) = z(i)/sqrt(h(i))
end do i
decl symm[series] r(nSeries,nSeries)
decl symm[series] q(nSeries,nSeries)
decl symm qMat(nSeries,nSeries) qMat1(nSeries,nSeries)
decl symm rMat(nSeries,nSeries) zMat1(nSeries,nSeries)
decl frml[symm] qf
nonlin(parmset=corrParms) alpha_c beta_c
*
* Define the q-function
frml qf = (zMat1 = %outerxx(%xt(z,t-1))-rMat),$
          (qMat1 = %xt(q,t-1)-rMat),$
          rMat + alpha_c*zMat1 + beta_c*qMat1
*
* Initialize values
VCV(MATRIX=rMat,NOPRINT)
# z
*
do i=1,nSeries
   do j=1,i
      set q(i,j) = rMat(i,j)
   end do j
end do i
*
*  Define correlation part of the log likelihood
FRML CLOGL = qMat = qf(t), zVec = %xt(z,t), %pt(q,t,qMat),$
    %pt(r,t,%mqform(qMat,inv(%diag(%sqrt(%xdiag(qMat)))))),$
     qMat = %xt(r,t),$
    %logdensity(qMat,zVec)
comp alpha_c = 0.05, beta_c = 0.80
MAXIMIZE(parmset=corrParms,$
  METHOD=SIMPLEX,ITERS=10) CLOGL GSTART+3 GEND
MAXIMIZE(parmset=corrParms,$
  METHOD=BFGS,ITERS=100) CLOGL GSTART+3 GEND
```

For further details regarding DCC, see Engle (2002) and Engle and Sheppard (2001) (Engle's home page).