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Approximation of certain stochastic integrals and
interpolation

(joint work with Anni Toivola)

1. Formulation of the problem

- $W = (W_t)_{t \in [0,1]}$ standard Brownian motion
- $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \in [0,1]})$ stochastic basis satisfying the usual conditions where the filtration is generated by W
- $S_t := e^{W_t - \frac{t}{2}}$ Geometric Brownian motion
- $f : (0, \infty) \rightarrow \mathbb{R}$ with $\mathbb{E}f(S_1)^2 < \infty$ (i.e. pay-off of European type option)

Martingale representation of $f(S_1)$: Define

$$F(t, x) := \mathbb{E}_{S_t=x} f(S_1) = \mathbb{E} f(xS_{1-t})$$

so that $F \in C^\infty([0, 1) \times (0, \infty))$,

$$\frac{\partial F}{\partial t} + \frac{x^2}{2} \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{and} \quad F(1, x) = f(x).$$

By Itô's-formula (\mathbb{P} -a.s.):

$$f(S_1) = \mathbb{E} f(S_1) + \int_0^1 \frac{\partial F}{\partial x}(t, S_t) dS_t \quad \mathbb{P}\text{-a.s.}$$

Riemann-approximation of $\int_0^t \frac{\partial F}{\partial x}(u, S_u) dS_u$: Consider $\tau = (t_i)_{i=0}^n$ with $0 = t_0 \leq \dots \leq t_n = 1$ and

$$C_t(f, \tau) := \int_0^t \frac{\partial F}{\partial x}(u, S_u) dS_u - \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, S_{t_{i-1}})(S_{t_i \wedge t} - S_{t_{i-1} \wedge t}).$$

- $\mathbb{E}f(S_1) + \int_0^1 \frac{\partial F}{\partial x}(t, S_t) dS_t$

Value of the **perfect** portfolio to hedge $f(S_1)$.

- $\mathbb{E}f(S_1) + \sum_{i=1}^n \frac{\partial F}{\partial x}(t_{i-1}, S_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})$

Value of the **discretely** adjusted portfolio to hedge $f(S_1)$.

- $C_t(f, \tau)$: Risk coming from the discretely adjusted portfolio.

Problem 1 Behavior of $(C_t(f, \tau))_{t \in [0,1]} \iff$ Properties of f ?

Is the problem interesting? Take equidistant nets $\tau_n := \left(\frac{i}{n}\right)_{i=0}^n$.

- *Zhang* 1998 : f non-linear absolutely continuous function with polynomially bounded derivative. Then

$$\|C_1(f, \tau_n)\|_{L_2} \sim n^{-\frac{1}{2}}.$$

- *Gobet-Temam* 2001 : Let $K > 0$ and $f(x) := \begin{cases} 1 & : x \geq K \\ 0 & : x < K \end{cases}$.

Then

$$\|C_1(f, \tau_n)\|_{L_2} \sim n^{-\frac{1}{4}}$$

but, with an independent Brownian motion \widetilde{W} ,

$$\sqrt{n}C_1(f, \tau_n) \rightarrow_D \widetilde{W} \frac{1}{2} \int_0^1 \left(S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) \right)^2 dt.$$

2. Weighted Besov spaces

- $d\gamma(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$ Gaussian measure on \mathbb{R}
- For $f \in C_p^\infty(\mathbb{R})$ (derivatives are polynomially bounded) we define

$$\|f\|_{D_{1,2}} := \sqrt{\|f\|_{L_2(\gamma)}^2 + \|f'\|_{L_2(\gamma)}^2}.$$

Definition 2 *The Sobolev-space $D_{1,2}(\gamma)$ is the completion of $[C_p^\infty(\mathbb{R}), \|\cdot\|_{D_{1,2}}]$.*

Interpolation spaces $(X_0, X_1)_{\theta, q}$ with $\theta \in (0, 1)$, $q \in [1, \infty]$

- X_0 and X_1 Banach spaces ($X_0 \subseteq X_1$ or $X_1 \subseteq X_0$)
- K -functional for $x \in X_0 + X_1$:

$$K(x, t; X_0, X_1) := \inf \left\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_i \in X_i \right\}.$$

- For $x \in X_0 + X_1$ we let

$$\|x\|_{(X_0, X_1)_{\theta, q}} := \left\| t^{-\theta} K(x, t; X_0, X_1) \right\|_{L_q \left((0, \infty), \frac{dt}{t} \right)}.$$

Definition 3 For $q \in [1, \infty]$ and $\theta \in (0, 1)$ we define the Besov spaces

$$B_{2, q}^{\theta}(\gamma) := (L_2(\gamma), D_{1, 2}(\gamma))_{\theta, q}.$$

- θ is the fractional smoothness
- q is a fine tuning parameter

Example 4 Let $K > 0$ and $\alpha \in (0, \frac{1}{2})$.

<i>function</i> $f(x)$	<i>fractional smoothness</i> θ in $B_{2,\infty}^\theta(\gamma)$
$\max\{x - K, 0\}$	1
$(\max\{x - K, 0\})^\alpha$	$\frac{1}{2} + \alpha$
$\chi_{[K,\infty)}(x)$	$\frac{1}{2}$
$h_\alpha\left(\frac{1}{2} + \log x\right)$	$\frac{1}{2} - \alpha$

$$h_\alpha(y) := \begin{cases} y^{-\alpha} & : y > 0 \\ 0 & : y \leq 0 \end{cases}$$

3. Approximation

Theorem 5 (Geiss-Hujo, 2004) For $\theta \in (0, 1]$ the following assertions are equivalent:

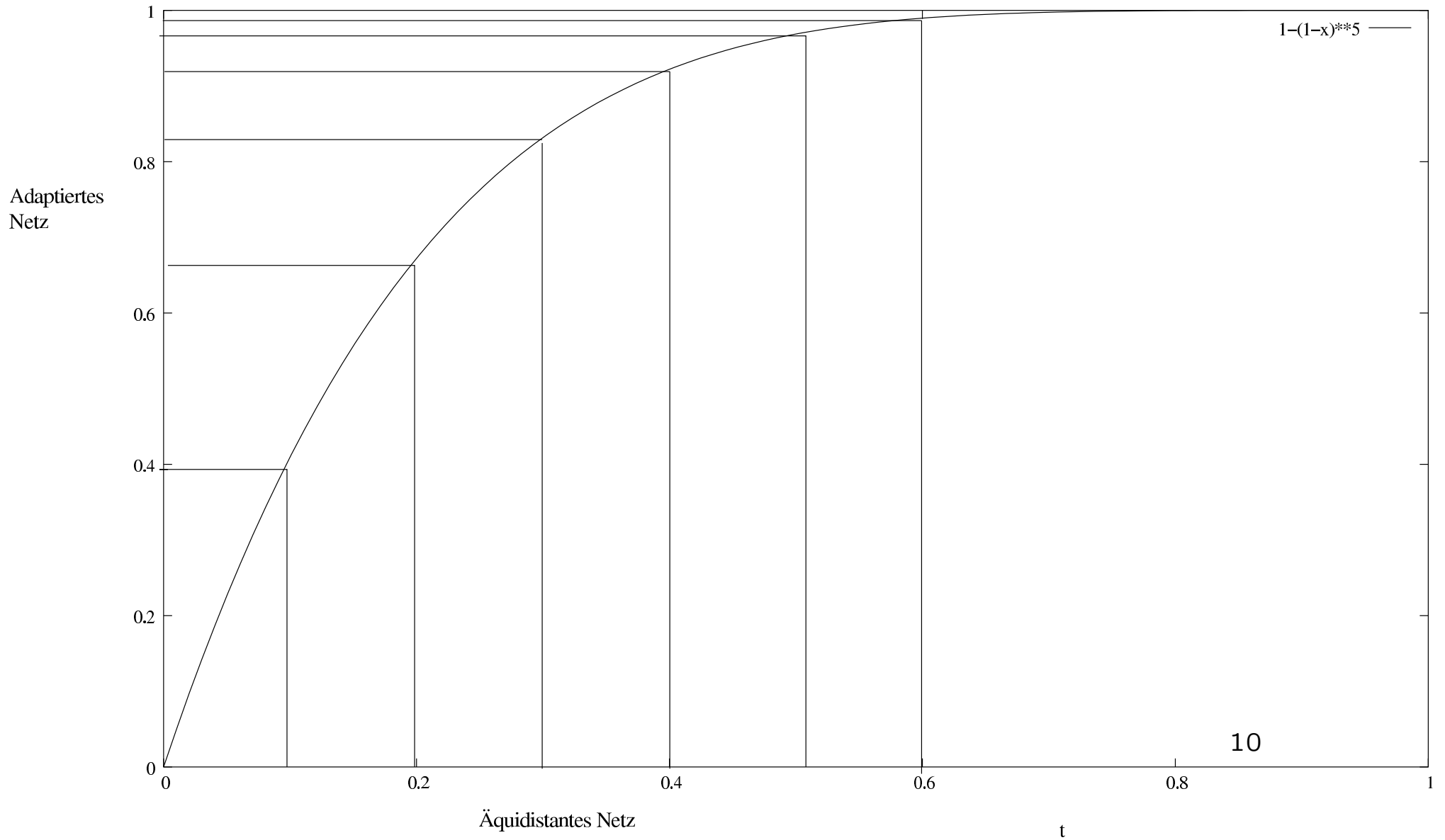
(1) Fractional smoothness: $f \left(e^{x-\frac{1}{2}} \right) \in B_{2,2}^\theta(\gamma)$.

(2) Optimal rate with special nets: There is a constant $c > 0$, so that

$$\|C_1(f, \tau_n^\theta)\|_{L_2} \leq \frac{c}{\sqrt{n}} \quad \text{where} \quad t_i^{(n,\theta)} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}}.$$

(3) $\int_0^1 (1-t)^{1-\theta} \mathbf{E} \left(S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) \right)^2 dt < \infty$.

$$h(t) := 1 - (1 - t)^{\frac{1}{\theta}} \text{ and } h\left(\frac{i}{n}\right) = t_i^{(n, \theta)}:$$



Following situation for $f(x) = \chi_{[K,\infty)}(x)$ and $\tau_n = (i/n)_{i=0}^n$:

- $\|C_1(\tau_n)\|_{L_2} \sim n^{-\frac{1}{4}}$.
- $n^{\frac{1}{2}} C_1(f, \tau_n) \rightarrow_D \widetilde{W} \frac{1}{2} \int_0^1 \left(S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) \right)^2 dt$
- $\|C_1(\tau_n^\theta)\|_{L_2} \sim n^{-\frac{1}{2}}$ for $0 < \theta < \frac{1}{2}$ since

$$\chi_{[K,\infty)}(x) \in B_{2,\infty}^{\frac{1}{2}}(\gamma) \subseteq B_{2,2}^\theta(\gamma).$$

Question: Do the non-equidistant nets τ_n^θ make sense in the context of weak convergence?

Observation: In the above example one has

$$\mathbb{E} \left| \widetilde{W} \frac{1}{2} \int_0^1 \left(S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) \right)^2 dt \right|^2 = \infty.$$

Theorem 6 (Geiss-Toivola) Let $f \left(e^{-\frac{1}{2}} \right) \in B_{2,2}^\theta(\gamma)$. Then

$$n^{\frac{1}{2}} C_1(f, \tau_n^\theta) \rightarrow_D \widetilde{W} \frac{1}{2} \int_0^1 \frac{1}{\theta} (1-u)^{1-\theta} \left[S_u^2 \frac{\partial^2 F}{\partial x^2}(u, S_u) \right]^2 du =: Y$$

with $\mathbb{E}|Y|^2 < \infty$ and

$$t_i^{(n,\theta)} := 1 - \left(1 - \frac{i}{n} \right)^{\frac{1}{\theta}}.$$

There is a converse statement which shows that Theorem 6 is basically optimal:

Proposition 7 (G-T) *Let $f(S_1) \in L_2$ and $\theta \in (0, 1]$. If*

$$\sup_{t \in [0,1)} \left\| D - \lim_n n^{\frac{1}{2}} C_t(f, \tau_n^\theta) \right\|_{L_2} < \infty,$$

then $f\left(e^{\cdot - \frac{1}{2}}\right) \in B_{2,2}^\theta(\gamma)$ where $t_i^{(n,\theta)} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}}$.

There is also a process version of Theorem 6:

Theorem 8 (G-T) *Let $f \left(e^{\cdot - \frac{1}{2}} \right) \in B_{2,2}^\theta(\gamma)$ and $T \in [0, 1)$. Then*

$$n^{\frac{1}{2}} C_{\psi_n^{-1} \left(\frac{s}{T} \psi_n(T) \right)} (f, \tau_n^\theta) \rightarrow_D \widetilde{W} \int_0^T \frac{1}{\theta} (1-u)^{1-\theta} \left[S_u^2 \frac{\partial^2 F}{\partial x^2} (u, S_u) \right]^2 du$$

in $C[0, T]$ with

$$t_i^{(n, \theta)} := 1 - \left(1 - \frac{i}{n} \right)^{\frac{1}{\theta}}$$

where

$$\psi_n(s) := \frac{1}{2\theta} \int_0^s (1-u)^{1-\theta} \left[S_u^2 \frac{\partial^2 F}{\partial x^2} (u, S_u) \right]^2 du.$$

Corollary 9 Let $f\left(e^{\cdot-\frac{1}{2}}\right) \in B_{2,2}^\theta(\gamma)$. Then

$$n^{\frac{1}{2}} \sup_{t \in [0,1]} |C_t(f, \tau_n^\theta)| \rightarrow_D \sup_{t \in [0,1]} \left| \widetilde{W} \frac{1}{2} \int_0^t \frac{1}{\theta} (1-u)^{1-\theta} \left[S_u^2 \frac{\partial^2 F}{\partial x^2}(u, S_u) \right]^2 du \right|$$

with

$$t_i^{(n,\theta)} := 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}}.$$

4. Idea for the proof of Theorem 6

Step 1: Decomposition of the error process: Let

$$t_i^n := t_i^{(n,\theta)} = 1 - \left(1 - \frac{i}{n}\right)^{\frac{1}{\theta}},$$

then

$$\begin{aligned}
& \sqrt{n}C_1(\tau^{n,\theta}) \\
= & \sqrt{n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left[\frac{\partial F}{\partial x}(u, S_u) - \frac{\partial F}{\partial x}(t_{i-1}^n, S_{t_{i-1}^n}) \right. \\
& \qquad \qquad \qquad \left. - \frac{\partial^2 F}{\partial x^2}(t_{i-1}^n, S_{t_{i-1}^n})(S_u - S_{t_{i-1}^n}) \right] dS_u \\
& + \sqrt{n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left[S_u - S_{t_{i-1}^n} \right] \frac{\partial^2 F}{\partial x^2}(t_{i-1}^n, S_{t_{i-1}^n})(S_u - S_{t_{i-1}^n}) dW_u \\
& + \sqrt{n} \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} S_{t_i^n} \frac{\partial^2 F}{\partial x^2}(t_{i-1}^n, S_{t_{i-1}^n})(S_u - S_{t_{i-1}^n}) dW_u \\
= & I_n^1 + I_n^2 + I_n^3.
\end{aligned}$$

Lemma 10 *One has that*

$$\sqrt{n} \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left[\frac{\partial F}{\partial x} (u, X_u) - \frac{\partial F}{\partial x} \left(t_{i-1}^n, S_{t_{i-1}^n} \right) - \frac{\partial^2 F}{\partial x^2} \left(t_{i-1}^n, S_{t_{i-1}^n} \right) \left(S_u - S_{t_{i-1}^n} \right) \right] dS_u \right\|_{L_2} \xrightarrow{n} 0$$

and

$$\sqrt{n} \left\| \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \left[S_u - S_{t_{i-1}^n} \right] \frac{\partial^2 F}{\partial x^2} \left(t_{i-1}^n, S_{t_{i-1}^n} \right) \left(S_u - S_{t_{i-1}^n} \right) dW_u \right\|_{L_2} \xrightarrow{n} 0.$$

Step 2: Reduction to $t \in (0, 1)$ to treat I_n^3 :

Lemma 11 *Let $\epsilon \in (0, 1)$ and $t \in (0, 1)$. Then, for $\lambda > 0$,*

$$\begin{aligned} \mathbb{P} \left(\sqrt{n}C_t(\tau^{n,\theta}) > \frac{\lambda}{1-\epsilon} \right) &\leq d(t) \frac{(1-\epsilon)^2}{\epsilon^2 \lambda^2} \\ &\leq \mathbb{P} \left(\sqrt{n}C_1(\tau^{n,\theta}) > \lambda \right) \\ &\leq \mathbb{P} \left(\sqrt{n}C_t(\tau^{n,\theta}) > \lambda(1-\epsilon) \right) + \frac{d(t)}{\epsilon^2 \lambda^2} \end{aligned}$$

where

$$d(t) := c_\theta \int_{\tilde{t}}^1 (1-s)^{1-\theta} E \left(S_s^2 \frac{\partial^2 F}{\partial x^2} (s, S_s) \right)^2 ds$$

and \tilde{t} is left-hand side endpoint of the interval $t \in [t_{i-1}, t_i)$. where

Step 3: Treatment of I_n^3 :

Theorem 12 (Rootzen) *Let ψ^n , $n = 1, 2, \dots$, be progressively measurable processes such that*

$$\sup_{0 \leq t \leq 1} \left| \int_0^t [\psi_s^n]^2 ds - \tau_t \right| \rightarrow_P 0 \quad \text{as } n \rightarrow \infty$$

for some adapted continuous process τ , and such that

$$\sup_{0 \leq t \leq 1} \left| \int_0^t \psi_s^n ds \right| \rightarrow_P 0 \quad \text{as } n \rightarrow \infty.$$

Then

$$\int_0^1 \psi_s^n dW_s \rightarrow_D \widetilde{W}_{\tau_T} \quad \text{as } n \rightarrow \infty$$

for an independent Brownian motion \widetilde{W} .

8. Related references.

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