Mathematics of Financial Derivatives*

Sessions 6-10

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Risk and Return

State Price: $Q(\omega)/B_1(\omega)$ ($B_1 = 1 + r_f$, r_f the risk free return, $B_0 = 1$)

State Price Density:

$$L(\omega) = \frac{Q(\omega)}{P(\omega)}, \tag{1}$$

$\omega \in \Omega$, Q the risk neutral probability measure and $P(\omega)$ the objective probability measure. $L(\omega)$ is also called the state price vector or some times state price deflator.

Remark: In statistical terms $L(\omega)$ defines the *likelihood ratio* between measures Q and P, and in more general situations it is known as *Radon-Nikodym derivative* of Q with respect to P.

Remark: The role of the objective probabilities is that they indicate which events are possible and which events are impossible.

The risk-neutral probabilities serve as tools for computing arbitrage-free derivative prices.

Remark: The ratio

$$M(\omega) = \frac{1}{B_1} L(\omega) \tag{2}$$

is called the *stochastic discount factor*.

For a contingent claim with pay-off $X(\omega)$ the arbitrage free price is

$$C = \frac{1}{B_1} \mathbb{E}_Q[X]$$

where $B_1 = 1 + r_f$ and $\mathbb{E}_Q$ denotes the expectation w.r.t the risk neutral probability Q.

It is straightforward to show that

$$C = \mathbb{E}[M\,X],$$

where $\mathbb{E}$ is the expectation w.r.t the objective probability P.

Return of an asset n:

(3) $$R_n = \frac{S_n(1) - S_n(0)}{S_n(0)},$$

$n = 1, \ldots, N$.

Bank account (generally)

$$R_0 = \frac{B_1 - B_0}{B_0} = r_f. \tag{4}$$

$S_n \geq 0 \Rightarrow R_n \geq -1$.

If Q is a risk neutral probability with $Q(\omega) > 0$ for all $\omega \in \Omega$,

$$\mathbb{E}_Q\left[\frac{R_n - R_0}{1 + R_0}\right] = 0 \tag{5}$$

for all $n = 1, \ldots, N$.

In particular if $R_0 = r_f$ is deterministic, (5) becomes

$$\mathbb{E}_Q[R_n] = r_f. \tag{6}$$

Risk premium: $\mathbb{E}[R_n(1) - r_f]$

Example: For simplicity, suppose $r_f = 0$. Assume $S(0) = 100$, $S(1) = 120$ with probability $p_1 = 0.6$ and $S(1) = 80$ with probability $p_2 = 0.40$.

The discounted expected price is then

$$\frac{1}{B_1}\mathbb{E}[S(1)] = 1 \cdot (0.6 \times 120 + 0.4 \times 80) = 104$$

thus markets seems to be risk averse and with 4 $(= 104 - 100)$ units risk premium.

Recall,

CAPM: $\mathbb{E}[R] - r_f = \lambda\,\mathbb{C}\text{ov}[R, R_{\texttt{mkt}}]$,

where R is a return of a single asset or a portfolio, λ is a constant, and $R_{\texttt{mkt}}$ is the return of the "market portfolio".

It proves to be also:

$$\mathbb{E}[R] - r_f = -\mathbb{C}\text{ov}(R, L). \tag{7}$$

This follows easily (solution given in class room) from the fact that expected returns with respect to the risk neutral density Q equal the risk free rate.

Example: (Continued) In terms of returns

$R_1(\omega_1) = 100 \times (120 - 100)/100 = 20\%$,

$R_1(\omega_2) = 100 \times (80 - 100)/100 = -20\%$,

and $\mathbb{E}[R] - r_f = 4\%$.

The risk neutral measure Q can be solved from

$$100 = \frac{1}{B_1}\mathbb{E}_Q[S] = 120q + 80(1-q),$$

which gives $q = 1/2$.

Thus,

$$L(\omega_1) = \frac{0.6}{0.5} = 1.2$$

and

$$L(\omega_2) = \frac{0.4}{0.5} = 0.8$$

$$\begin{aligned}\mathbb{C}\text{ov}[R, L] &= \mathbb{E}[(L-1)(R-4)] \\ &= (1.2-1) \times (20-4) \times 0.6 \\ &\quad +(0.8-1) \times (-20-4) \times 0.4 \\ &= -4\end{aligned}$$

i.e.,

$$\mathbb{E}[R] - r_f = -\mathbb{C}\text{ov}[R, L].$$

Remark: The risk premium can be characterized also as

$$E[R] - r_f = \mathbb{E}[R] - \mathbb{E}_Q[R], \tag{8}$$

because $\mathbb{E}[R] - \mathbb{E}_Q[R] = -\mathbb{C}\mathrm{ov}[R, L]$.

Example: (Continued) Suppose there is a call option with exercise price 110.

Arbitrage free price

$$\begin{aligned} C &= \tfrac{1}{1+r_f}\mathbb{E}_Q[\max(S-110,0)] \\ &= \tfrac{1}{1+0}(10\times 0.5+0\times 0.5)=5 \end{aligned}$$

Risk premium in the arbitrage free price

$\mathbb{E}[\max(S-110,0)]-\mathbb{E}_Q[\max(S-110,0)]=6-5=1$,

or $100\times 1/5=20\%$.

Porfolio strategy: $H = (H_0, H_1, \ldots, H_N)$, where H_n is the number of shares (can be negative), $n = 0, 1, \ldots, N$.

Value:

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t) \tag{9}$$

$t = 0, 1$

Gain: $G = V_1 - V_0$

$$G = H_0 B_0 R_0 + \sum_{n=1}^{N} H_n S_n(0) R_n. \tag{10}$$

Remark: $H_n S_n(0)$ is the amount invested in security n at $t = 0$.

Return of the portfolio

$$R = \frac{G}{V_0} = \frac{H_0}{V_0} r_f + \sum_{n=1}^{N} \left[\frac{H_n S_n(0)}{V_0} \right] R_n. \tag{11}$$

Then again (7) holds.

Beta of a trading strategy (portfolio):

Suppose a contingent claim with end value $V_1' = a + bL$, where a and $b \neq 0$ are scalars, is attainable (denote the strategy as H').

Now $R' = (V_1' - V_0')/V_0'$, or $V_0'(1 + R') = a + bL$

Then $\mathbb{C}\text{ov}[R, L] = \frac{V_0'}{b}\mathbb{C}\text{ov}[R, R']$ and thus

(12) $$\mathbb{E}[R] - r_f = -\frac{V_0'}{b}\mathbb{C}\text{ov}[R, R'].$$

In particular, if we choose $H = H'$,

$$\mathbb{E}[R'] - r_f = -\frac{V_0'}{b}\mathbb{V}\mathrm{ar}[R'],$$

or

$$\frac{\mathbb{E}[R'] - r_f}{\mathbb{V}\mathrm{ar}[R']} = -\frac{V_0'}{b},$$

which gives from (12)

(13) $$\mathbb{E}[R] - r_f = \beta\left(\mathbb{E}[R'] - r_f\right),$$

where

(14) $$\beta = \frac{\mathbb{C}\mathrm{ov}[R, R']}{\mathbb{V}\mathrm{ar}[R']}$$

is called the beta of the trading strategy H with respect to trading strategy H' (c.f. CAPM).

Remark: With a deterministic interest rate r_f and arbitrary scalars a and $b \neq 0$, the contingent claim $V' = a + bL$ is attainable iff the state price density L is.

This is because

$$H_0(1 + r_f) + \sum_{n=1}^{N} H_n S_n(1) = a + bL$$

iff

$$\frac{1}{b}\left[H_0 - \frac{a}{1 + r_f}\right](1 + r_f) + \sum_{n=1}^{N} \frac{1}{b} H_n S_n(1) = L,$$

in which case the trading strategy is $H' = (H'_0, H'_1, \ldots, H'_n)$ with

$$H'_0 = \frac{1}{b}\left[H_0 - \frac{a}{1 + r_f}\right]$$

and

$$H'_n = \frac{1}{b} H_n.$$

Single Period Consumption and Investment*

Here we deal with relations of optimal investments and consumption-investment to no arbitrage and risk neutral probability measure.

Optimal portfolios and viability

Problem of choosing the best trading strategy:

$$H = (H_0, H_1, \ldots, H_N) \in \mathbb{R}^{N+1}.$$

Utility function; $u : \mathbb{R} \times \Omega \to \mathbb{R}$.

$w \to u(w, \omega)$ differentiable, concave and strictly increasing for each $\omega \in \Omega$.

Remark: The utility function u can depend on the terminal wealth w and state ω. However, mostly it suffices for u to depend only on the terminal wealth, implying that u is a concave function with a single argument.

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Expected utility of terminal wealth:

$$\mathbb{E}\left[u(V_1)\right] = \sum_{\omega \in \Omega} u\left(V_1(\omega), \omega\right) P(\omega). \tag{15}$$

Objective (optimal portfolio problem)

$$\begin{aligned} &\max{}_H \, \mathbb{E}\left[u(V_1)\right] \\ &\text{s.t} \quad V_0 = \nu, \end{aligned} \tag{16}$$

where ν is the fixed initial wealth.

Because $V_1 = B_1 V_1^*$ and $V_1^* = V_0^* + G^*$, (16) is equivalent to

$$\max_H \mathbb{E}\left[u(B_1\{\nu + H_1 \Delta S_1^* + \cdots + H_N \Delta S_N^*\})\right]. \tag{17}$$

Results: If there is a solution to (17) then there are no arbitrage opportunities, or equivalently that there exist a RNPM $Q(\omega)$.

This is, because if there exist an arbitrage strategy $\widehat{H}$ and H is a solution to (17), then strategy $\tilde{H} = \widehat{H} + H$ (prove it) would be better than H, which is a contradiction.

Traditional solution:

Foc:

(18) $$\frac{\partial}{\partial H_n}\mathbb{E}\left[u\left(B_1\left\{\nu+\sum_{k=1}^{N}H_k\Delta S_k^*\right\}\right)\right]=0,$$

for all $n = 1, \ldots, N$, i.e.,

(19)

$$\sum_{\omega\in\Omega}P(\omega)u'\left(B_1\left\{\nu+\sum_{k=1}^{N}H_k\Delta S_k^*(\omega)\right\},\omega\right)B_1\Delta S_n^*(\omega)=0,$$

for all $n = 1, \ldots, N$, where

$$u'(x,\omega)=\frac{\partial u(x,\omega)}{\partial x}.$$

In short (19) becomes

(20) $$\mathbb{E}\left[B_1 u'(V_1)\Delta S_n^*\right]=0$$

for all $n = 1, \ldots, N$.

Recall: For $L(\omega) = Q(\omega)/P(\omega)$

$$\mathbb{E}[L\,\Delta S_n^*] = 0$$

for all $n = 1, \ldots, N$.

Thus,

$$Q(\omega) = \frac{B_1 P(\omega) u'\left(V_1(\omega), \omega\right)}{\mathbb{E}[B_1 u'(V_1)]} \tag{21}$$

defines a RNPM.

Thus, in particular if $B_1 = 1 + r_f$ is constant,

$$L(\omega) = \frac{Q(\omega)}{P(\omega)} = \frac{u'\left(V_1(\omega), \omega\right)}{\mathbb{E}[u'(V_1)]}, \tag{22}$$

i.e., the state price density is proportional to the marginal utility of terminal wealth.

What about the converse: If Q exists, does the optimal portfolio problem have a solution?

Answer: not for all utility functions.

We say that the model is *viable* if there exist a function $u : \mathbb{R} \times \Omega \to R$ and an initial wealth ν such that $w \to u(w, \omega)$ is concave and strictly increasing for each $\omega \in \Omega$ and such that the optimal portfolio problem (16) has a solution H.

Then:

The securities market model is viable iff there exist a RNPM Q.

Risk neutral computation approach:

Solving the optimal portfolio problem (16) can be difficult.

Utilizing RNPM, Q, is more efficient way.

Step 1: Define the set of contingent claims (i.e. random variables)

$$\mathbb{W}_\nu = \left\{W : \mathbb{E}_Q[W/B_1] = \nu\right\}. \tag{23}$$

Then we observe that $V_1 \in \mathbb{W}_\nu$. The subset $\mathbb{W}_\nu$ is called the *set of attainable wealths*

Step 2: Find a replicating portfolio that replicates the contingent claim that has the same payoff as the optimal portfolio strategy.

With this approach the optimization problem becomes

$$\text{(24)} \qquad \max_W \mathbb{E}[u(W)] \quad \text{s.t. } W \in \mathbb{W}_\nu.$$

Lagrange multiplier (λ) method:

$$\text{(25)} \qquad \max_{W,\lambda} \left\{ \mathbb{E}[u(W)] - \lambda \mathbb{E}_Q[W/B_1] \right\},$$

Now $\mathbb{E}_Q[W/B_1] = \mathbb{E}[LW/B_1]$, where $L = Q/P$.

Thus the difference in (25) becomes

(26)

$$\mathbb{E}\left[(u(W) - \lambda LW/B_1)\right]$$

$$= \sum_{\omega\in\Omega} P(\omega) \left\{u(W(\omega)) - \lambda L(\omega)W(\omega)/B_1(\omega)\right\}$$

Foc:

(27) $$u'(W(\omega)) = \lambda L(\omega)/B_1(\omega)$$

for all $\omega \in \Omega$.

This gives

(28) $$W(\omega) = I\left(\lambda L(\omega)/B_1(\omega)\right),$$

where I is the inverse function of u'.

λ must be then selected such that $W \in \mathbb{W}_\nu$, i.e.,

(29) $$\mathbb{E}_Q\left[I\left(\lambda L/B_1\right)/B_1\right] = \nu$$

is satisfied.

Example: (Pliska p.39)

$K = 3$, $N = 2$, $P(\omega_1) = 1/2$, and $P(\omega_2) = P(\omega_3) = 1/4$.
$r_f = 1/9$, $B_1 = 10/9$.

n	$S_n^*(0)$	$S_n^*(1)$		
		ω_1	ω_2	ω_3
1	6	6	8	4
2	10	13	9	8

Solving system of equations ($n = 1, 2$)

$S_n^*(0) = Q(\omega_1)S_n^*(1,\omega) + Q(\omega_2)S_n^*(1,\omega_2) + Q(\omega_3)S_n^*(1,\omega_3)$

together with $Q(\omega_1) + Q(\omega_2) + Q(\omega_3) = 1$ gives

$$Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = 1/3.$$

Thus, $L(\omega_1) = 2/3$, $L(\omega_2) = L(\omega_3) = 4/3$.

Suppose $u(w) = -e^{-w}$, such that $u'(w) = e^{-w} \equiv i$ and $w = -\log(i)$

Now

$\mathbb{E}[I(\lambda L/B_1)] = -\log\lambda - \mathbb{E}_Q[\log(L/B_1)] = -\log\lambda + 0.04873$

From (29)

$$(-\log\lambda + 0.04873)/B_1 = \nu$$

or ($B_1 = 10/9$)

$$\lambda = e^{0.04873 - \nu\times 10/9}.$$

Thus the optimal random variable is from (28)

$$\begin{aligned} W &= -\log(\lambda L(\omega)/B_1(\omega)) \\ &= \nu \times 10/9 - 0.04873 - \log(L(\omega)/B_1) \\ &= \begin{cases} (10/9)\nu + 0.46209, & \omega = \omega_1 \\ (10/9)\nu - 0.23105, & \omega = \omega_2, \omega_3 \end{cases} \end{aligned}$$

Consumption Investment Problem

Consumption process:

$$C = (C_0, C_1), \tag{30}$$

where C_0 a non-negative constant (consumption now, $t = 0$) and C_1 non-negative random variable (next period, $t = 1$, consumption).

Consumption-investment plan

$$(C, H), \tag{31}$$

where C is a consumption process and H is a trading strategy.

A consumption-invest plan is *admissible* if (1) $C_0 + V_0 = \nu$ ($\nu \geq 0$), the initial wealth at $t = 0$, and (2) $C_1 = V_1$.

How to check whether a given (C, H) is admissible?

Result 2.15: Assuming $\nu \geq 0$ and that C is fixed. Then there exist a trading strategy H such that (C, H) is admissible iff

$$C_0 + \mathbb{E}_Q[C_1/B_1] = \nu$$

for every RNPM Q.

Consumption investment problem:

Utility function $u : \mathbb{R}_+ \to \mathbb{R}$ concave, strictly increasing, and differentiable ($\mathbb{R}_+ = [0, +\infty)$).

(C, H) admissible

$$\text{(32)} \qquad \begin{aligned} &\max \{u(C_0) + \mathbb{E}[u(C_1)]\} \\ &\text{s.t.} \quad C_0 + V_0 = \nu \\ &\qquad\; C_1 = V_1, \\ &\qquad\; C_0 \geq 0, C_1 \geq 0 \end{aligned}$$

Recall

$$\text{(33)} \qquad V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t), \qquad t = 0, 1$$

$(B_0 = 1)$

Using (33) in (32):

Foc

$$u'(C_0) = \mathbb{E}[u'(C_1)B_1] \tag{34}$$
$$u'(C_0)S_n(0) = \mathbb{E}[u'(C_1)S_n(1)],$$
$$n = 1, \dots, N.$$

Again

$$S_n(0) = \mathbb{E}\left[\frac{B_1 u'(C_1)}{u'(C_0)} S_n^*(1)\right], \tag{35}$$

where $S_n^*(1) = S_n(1)/B_1$ discounted price, which implies the risk neutral probability measure

$$Q(\omega) = P(\omega)B_1 \frac{u'(C_1(\omega))}{u'(C_0)} \tag{36}$$

which is a risk neutral probability measure [$\sum_{\omega \in \Omega} Q(\omega) = 1$ due to the first equation in (34)]

Assuming complete markets, using risk neutral computation, the problem is again to identify the contingent claim C_1 that maximizes the utility and then find the replicating portfolio, which then is the solution of (32).

That is

$$
\begin{aligned}
& \max_{C_1} \{u(C_0) + \mathbb{E}[u(C_1)]\} \\
(37) \qquad & s.t \quad C_0 + \mathbb{E}_Q[C_1/B_1] = \nu \\
& \qquad\quad C_0 \geq 0, \;\; C_1 \geq 0
\end{aligned}
$$

Note that $V_0 = \mathbb{E}_Q[C_1/B_1]$.

Lagrange

(38)

$$\max_{C_1} u(C_0) + \mathbb{E}[u(C_1)] - \lambda \{C_0 + \mathbb{E}[LC_1/B_1]\}$$

(recall $\mathbb{E}_Q[C_1/B] = \mathbb{E}[LC_1/B_1]$).

Foc:

$$u'(C) = \lambda \quad \text{and} \quad u'(C_1(\omega)) = \lambda L(\omega)/B_1.$$

(39) $C_0 = I(\lambda)$ and $C_1(\omega) = I(\lambda L(\omega)/B_1)$,

where $I(\cdot)$ is the inverse of $u'(\cdot)$ and λ is such that the constraint in (37) is satisfied, i.e.,

(40) $$I(\lambda) + \mathbb{E}_Q [I(\lambda L/B_1)/B_1] = \nu.$$

Example: $u(c) = \log(c)$, then $u'(c) = 1/c$, $I(i) = 1/i$. Thus, from (39)

$$C_0 = 1/\lambda \text{ and } C_1(\omega) = \frac{B_1}{\lambda L(\omega)}$$

and form (40)

$$\frac{1}{\lambda} + \frac{1}{\lambda}\mathbb{E}_Q[1/L] = \frac{2}{\nu}$$

i.e.,

$$\frac{2}{\lambda} = \nu \text{ or } \lambda = \frac{2}{\nu}$$

and thus

$$C_0 = \nu/2$$

and

$$C_1(\omega) = \frac{\nu B_1}{2L(\omega)} = \frac{\nu B_1 P(\omega)}{2Q(\omega)}.$$

Note that in the second equation above we have used the fact that $\mathbb{E}_Q[1/L] = \mathbb{E}[L(1/L)] = \mathbb{E}[1] = 1$.

Remark: The problem can be easily extended to

(41) $$\max_{C_1} \{u(C_0) + \beta\mathbb{E}[u(C_1)]\},$$

where $0 < \beta \leq 1$ is a discount factor of consumption (time value of deferred consumption).

Mean-Variance Portfolio Analysis*

Assume deterministic risk-free rate r_f, no arbitrage opportunities, and a portfolio exists such that $\mathbb{E}[R] \neq r_f$.

Mean-variance portfolio problem:

$$(42) \qquad \begin{aligned} &\min \mathbb{V}\mathrm{ar}[R] \\ &\text{s.t. } \mathbb{E}[R] = \rho \end{aligned}$$

where ρ is a specified scalar and R is a portfolio return such that

$$(43) \qquad R = \frac{H_0}{V_0} r_f + \sum_{n=1}^{N} \left[\frac{H_n S_n(0)}{V_0} \right] R_n.$$

The feasible solutions of (42) are those of $\rho \geq r_f$, which is non-empty because of the assumption of the existence of a portfolio with $\mathbb{E}[R] \neq r_f$.

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Set $F_n = H_n S_n(0)/V_0$, the starting period portfolio weights with

(44) $$\sum_{n=0}^{N} F_n = 1.$$

Result: R is a portfolio return iff

(45) $$R = (1 - F_1 - \ldots - F_N) r_f + \sum_{n=1}^{N} F_n R_n.$$

The portfolio problem becomes:

(46)
$$\min F \mathbb{C} F'$$

$$\text{s.t. } (1 - F_1 - \ldots - F_N) r_f + \sum_{n=1}^{N} F_n \mathbb{E}[R_n] = \rho$$

where $F = (F_1, F_2, \ldots, F_N)'$ and $\mathbb{C}$ is the $N \times N$ return covariance matrix.

Again we apply here the risk neutral computation approach.

Consider the problem

$$\begin{aligned} & \min \mathbb{V}\mathrm{ar}[V_1] \\ & \text{s.t. } \mathbb{E}[V_1] = V_0(1+\rho) \end{aligned} \tag{47}$$

where $V_0 = \nu > 0$ is a given constant.

Result: V_1 is a solution of (47) iff $R = (V_1 - V_0)/V_0$ is the solution of (46).

Using Lagrange, the objective function for (47) becomes

$$\min \mathbb{V}\text{ar}[V_1] - \beta \mathbb{E}[V_1] \tag{48}$$

where β is the Lagrange multiplier and

$$\mathbb{E}[V_1] = V_0(1 + \rho).$$

Because for any random variable U for which $\mathbb{E}[U] = \mathbb{E}[V]$, $\mathbb{V}\mathrm{ar}[V] \leq \mathbb{V}\mathrm{ar}[U]$ implies $E[V^2] \leq \mathbb{E}[U^2]$, problem (48) can be written as

$$\text{(49)} \qquad \max \mathbb{E}\left[-\frac{1}{2}V_1^2 + \beta V_1\right] \text{ s.t. } V_0 = \nu.$$

Using the risk neutral computation approach, the solution is

$$\text{(50)} \qquad \hat{V} = \frac{\beta}{\mathbb{E}_Q[L]}\left(\mathbb{E}_Q[L] - L\right) + \nu(1 + r_f)\frac{L}{\mathbb{E}_Q[L]}.$$

Remark: $\mathbb{E}_Q[L] \geq 1$.

Because

(51)
$$\mathbb{E}\left[\hat{V}\right] = \frac{\beta}{\mathbb{E}_Q[L]}(\mathbb{E}_Q[L] - 1) + \nu(1 + r_f)\frac{1}{\mathbb{E}_Q[L]}$$

we have $\mathbb{E}\left[\hat{V}\right] = \nu(1+\rho)$ iff

(52)
$$\beta = \frac{\nu\left[(1+\rho)\mathbb{E}_Q[L] - (1+r_f)\right]}{\mathbb{E}_Q[L] - 1}.$$

Thus:

Result: Portfolio problems (47) and (49) are equivalent provided ρ and β are related according to (52).

It is notable that if $\rho = r_f$ then $\beta = \nu(1 + r_f)$.

Problem (49) is the standard utility maximization problem if the investor's utility function is quadratic,

$$u(w) = -\frac{1}{2}w^2 + \beta w.$$

In such a case (complete markets) using (51) with β from (52), the return of the optimal mean-variance portfolio is related to the state price density as follows

$$\hat{R} = \frac{\rho \mathbb{E}_Q[L] - r_f}{\mathbb{E}_Q[L] - 1} - \frac{\rho - r_f}{\mathbb{E}_Q[L] - 1} L. \qquad (53)$$

That is:

Result: In complete markets the return of the optimal mean-variance portfolio is an affine function of the state price density.

Finally, using this with the results of the risk-premium section:

Result: CAPM: If R_m is a solution of the mean-variance portfolio problem (42) for $\rho \geq r_f$ and if R is a return of an arbitrary portfolio, then

$$\mathbb{E}[R] - r_f = \frac{\mathbb{C}\mathrm{ov}[R, R_m]}{\mathbb{V}\mathrm{ar}[R_m]} \big(\mathbb{E}[R_m] - r_f\big). \tag{54}$$

This implies the two-fund separation principle (theorem):

Mutual fund principle: Given a fixed portfolio whose return is a solution of the mean-variance portfolio problem (42) corresponding to some mean return $\widehat{\rho} > r_f$. Then the solution of (42) can be achieved for any other mean return by a portfolio consisting of investments in just the riskless security and the fixed portfolio.