UMEÅ UNIVERSITET – DEPARTMENT OF COMPUTER SCIENCE

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II.3 Show that Yager's $\wedge$ and $\vee$ are, respectively, t-norms and co-t-norms.

Solution:

A) Yager's $\wedge$ is a t-norm;

"unit element"
$a \wedge 1 = 1 - min\{[(1-a)^p + 0]^{1/p}, 1\} = 1 - min\{(1-a), 1\} = 1 - (1-a) = a$

"monotonicity"
$a \geq b \rightarrow$
$a \wedge c = 1 - min\{[(1-a)^p + (1-c)^p]^{1/p}, 1\} \geq 1 - min\{[(1-b)^p + (1-c)^p]^{1/p}, 1\} = b \wedge c$

"commutativity"
$a \wedge b = 1 - min\{[(1-a)^p + (1-b)^p]^{1/p}, 1\} = 1 - min\{[(1-b)^p + (1-b)^p]^{1/p}, 1\} = b \wedge a$

"associativity"

$$a \wedge (b \wedge c) = 1 - min\left\{\left[(1-a)^p + \left(1 - \left(1 - min\{[(1-b)^p + (1-c)^p]^{1/p}, 1\}\right)\right)^p\right]^{1/p}, 1\right\} =$$

$$= 1 - min\{[(1-a)^p + min\{(1-b)^p + (1-c)^p, 1\}]^{1/p}, 1\} =$$

cases $(1-b)^p + (1-c)^p < 1$ and $(1-b)^p + (1-c)^p \geq 1$, are covered in the reduced expression

$$= 1 - min\{[(1-a)^p + (1-b)^p + (1-c)^p]^{1/p}, 1\} = (a \wedge b) \wedge c$$

Yager's $\vee$ is a co-t-norm;

"unit element"
$a \vee 0 = min\{[a^p + 0]^{1/p}, 1\} = min\{a, 1\} = a$

"monotonicity"
$a \geq b \rightarrow$
$a \vee c = min\{[a^p + c^p]^{1/p}, 1\} \geq min\{[b^p + c^p]^{1/p}, 1\} = b \vee c$

"commutativity"
$a \vee b = min\{[a^p + b^p]^{1/p}, 1\} = min\{[b^p + a^p]^{1/p}, 1\} = b \vee a$

"associativity"

$$a \vee (b \vee c) = min\left\{\left[a^p + min\{[b^p + c^p]^{1/p}, 1\}^p\right]^{1/p}, 1\right\} =$$

$$= min\{a^p + min\{[b^p + c^p], 1\}, 1\} = (a \vee b) \vee c$$

cases $b^p + c^p < 1$ and $b^p + c^p \geq 1$, are covered in the reduced expression

$$= min\{[a^p + b^p + c^p]^{1/p}, 1\} = (a \vee b) \vee c$$

II.4 Prove that for any t-norm T and any co-t-norm S we have

$$T(a,b) \leq \min(a,b) \ \textit{and} \ \max(a,b) \leq S(a,b).$$

Solution:

Let $a \leq b$, and use the "monotonicity" of the norms. Then for t-normas we have

$$T(a,b) \leq T(a,1) = a = \min(a,b)$$

and similarly for co-t-norms

$$\max\,(a,b) = b = S(0,b) \leq S(a,b)$$

II.6 Let $f(x) = x^2$ and let $A \in F$ be a symmetric triangular fuzzy number with membership function

$$A(x) = \begin{cases} 1 - |a - x|/\alpha & \text{if } |a - x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then use the extension principle to calculate the membership function of fuzzy set $f(A)$.

Solution:

The extension principle states that

$$F(A)(y) = \begin{cases} sup_{x|f(x)=y} A(x) & \text{if } \{\mathrm{x} \in \mathrm{X} | f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

In our case, supremum can be replaced by the maximum of two cases, as $f(x) = \mathrm{x}^2$ is a second order polynomial. From the symmetry and geometry of the functions involved, it can be seen that the maximum $A(x)$is found by the positive (negative) root of $f(x)$ for positive (negative) values of *a*. Hence $F(A)$ can be expressed as

$$F(A)(y) = \begin{cases} 1 - \left|a - sign(a^+) \cdot \sqrt{y}\right|/\alpha & \text{if } \left|a - sign(a^+)\sqrt{y}\right| \leq \alpha, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

II.7 Prove that $G_0(-;\alpha_1,\beta_1) \oplus G_0(-;\alpha_2,\beta_2) = G_0(-;\alpha_1+\alpha_2,\beta_1+\beta_2)$.

Solution:

Let $\mu_{A1} = G_0(-;\alpha_1;\beta_1)$ and $\mu_{A2} = G_0(-;\alpha_2;\beta_2)$. Now, using the extension principle, we get

$$\oplus\,(\mu_{A1},\mu_{A2}) = sup_{x_1+x_2=y}\min\{\mu_{A1}(x_1),\mu_{A2}(x_2)\}$$

It can be seen that the maximum value of $\min\{\mu_{A1}(x_1),\mu_{A2}(x_2)\}$ for $x_1+x_2=y$ is found when

$$\mu_{A1}(x_1) = \mu_{A2}(x_2) \Leftrightarrow \frac{x_1-\alpha_1}{\beta_1} = \frac{x_2-\alpha_2}{\beta_2} \Leftrightarrow x_2 = \frac{(x_1-\alpha_1)\beta_2}{\beta_1} + \alpha_2\,, \qquad (*)$$

yielding

$$= e^{-\frac{1}{2}\left(\frac{x_1-\alpha_1}{\beta_1}\right)^2} \qquad \text{(multiply by } (\beta_2+\beta_2))$$

$$= e^{-\frac{1}{2}\left(\frac{(\beta_2+\beta_2)x_1-(\beta_2+\beta_2)\alpha_1}{(\beta_2+\beta_2)\beta_1}\right)^2} \qquad \text{(rearrange, add } \alpha_2-\alpha_2)$$

$$= e^{-\frac{1}{2}\left(\frac{x_1+\frac{(x_1-\alpha_1)\beta_2}{\beta_1}+\alpha_2-\alpha_2-\alpha_1}{(\beta_2+\beta_2)}\right)^2} \qquad \text{(use (*))}$$

$$= e^{-\frac{1}{2}\left(\frac{(x_1+x_2)-(\alpha_2+\alpha_1)}{(\beta_2+\beta_2)}\right)^2}$$

$$= e^{-\frac{1}{2}\left(\frac{y-(\alpha_1+\alpha_2)}{(\beta_1+\beta_2)}\right)^2}$$

$$= G_0(y;\alpha_2+\alpha_1;\beta_2+\beta_2).$$

III.1 Given the rule base

IF x is $SMALL$ AND y is $SMALL$ AND z is $SMALL$ THEN $u = 0$
IF x is $MEDIUM$ AND y is $MEDIUM$ AND z is $MEDIUM$ THEN $u = 1$
IF x is BIG AND y is BIG AND z is BIG THEN $u = 2$
IF x is $SMALL$ AND y is $SMALL$ AND z is $MEDIUM$ THEN $u = 3$
IF x is $SMALL$ AND y is $SMALL$ AND z is BIG THEN $u = 4$
IF x is $SMALL$ AND y is $MEDIUM$ AND z is BIG THEN $u = 5$

where

$$SMALL(t) = \begin{cases} 1 - \frac{t}{4}, & \text{if } 0 \leq t \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$MEDIUM(t) = \begin{cases} 1 - \frac{|t-2|}{2}, & \text{if } 0 \leq t \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$BIG(t) = \begin{cases} 1 - \frac{4-t}{4}, & \text{if } 0 \leq t \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

use the Takagi-Sugeno method to compute the output for $x = 2$, $y = 2$ and $z = 4$.

Solution:

Compute, for each output output u_i, the corresponding activation level α_i. The activation levels are

$$\alpha_i = min(A_{i1}(x), A_{i2}(x), A_{i3}(x)),$$

where each $A_{ij}(t)$ is one of $SMALL(t)$, $MEDIUM(t)$ or $BIG(t)$. From top to bottom above we have

$u_1 = 0,\quad \alpha_1 = 0$
$u_2 = 1,\quad \alpha_2 = 0$
$u_3 = 2,\quad \alpha_3 = 0.5$
$u_4 = 3,\quad \alpha_4 = 0$
$u_5 = 4,\quad \alpha_5 = 0.5$
$u_6 = 5,\quad \alpha_6 = 0.5$

The final control value becomes

$$u = \frac{0.5 \cdot 2 + 0.5 \cdot 4 + 0.5 \cdot 5}{0.5 + 0.5 + 0.5} = \frac{5.5}{1.5} \approx 3.667$$

III.2 Given the rule base

IF x is $SMALL$ AND y is $SMALL$ AND z is $SMALL$ THEN u is $SMALL$
IF x is $MEDIUM$ AND y is $MEDIUM$ AND z is $MEDIUM$ THEN u is $MEDIUM$
IF x is BIG AND y is BIG AND z is BIG THEN $u = 2$ is BIG
IF x is $SMALL$ AND y is $MEDIUM$ AND z is BIG THEN u is $MEDIUM$

where

$$SMALL(t) = \begin{cases} 1 - \frac{t}{4}, & \text{if } 0 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$MEDIUM(t) = \begin{cases} 1 - \frac{|t-2|}{2}, & \text{if } 0 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$BIG(t) = \begin{cases} 1 - \frac{4-t}{4}, & \text{if } 0 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

use the Mamdani method of inference together with a defuzzification method of your choice to compute the output of inputs given by $x = 2$, $y = 2$ and $z = 4$.

Solution:

Compute, for each output output u_i, the corresponding activation level α_i. The activation levels are

$$\alpha_i = \bigwedge_{j=1}^{3} A_{ij}(x_j)$$

where each $A_{ij}(t)$ is one of $SMALL(t)$, $MEDIUM(t)$ or $BIG(t)$. From top to bottom above we have

$u_1 = SMALL, \quad \alpha_1 = 0$
$u_2 = MEDIUM, \quad \alpha_2 = 0$
$u_3 = BIG, \quad \alpha_3 = 0.5$
$u_4 = MEDIUM, \quad \alpha_4 = 0.5$

From this the following conclusion fuzzy set is derived

$$U(t) = \bigvee_{i=1}^{4} (\alpha_i \wedge u_i) = \begin{cases} 0.5t & 0 \le t \le 1 \\ 0.5 & 1 < t \le 4 \end{cases}$$

Using the CoG defuzzification the final control value becomes

$$U = \frac{\int_0^4 t \cdot U(t)dt}{\int_0^4 U(t)dt} = \frac{\int_0^1 0.5t^2\ dt + \int_1^4 0.5t\ \ dt}{\int_0^1 0.5t\ dt + \int_1^4 0.5\ \ dt} = \frac{47/12}{7/4} \approx 2.238$$