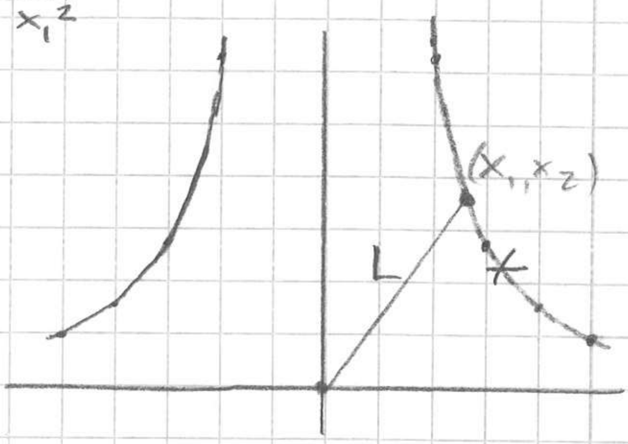
MATH2040 Optimoinnin erikoiskurssi — Kevät 2010

Harjoitus 5 — 10.2.2010

Nämä harjoitukset liittyvät monen muuttujan yhtälörajoitettuun optimointiin (WW osio 11.8).

1. Etsi tasokäyrän $\{(x_1, x_2); x_1^2 x_2 = 25\}$ piste, joka on lähinnä origoa.

$$x_1^2 x_2 = 25 \iff x_2 = \frac{25}{x_1^2}$$



$$\begin{cases} \min \sqrt{x_1^2 + x_2^2} \\ \text{ehdolla } x_1^2 x_2 = 25 \end{cases}$$

$$\sim \begin{cases} \min x_1^2 + x_2^2 \\ \text{ehdolla } x_1^2 x_2 = 25 \end{cases}$$

Lagrangen funktio

$$L(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda(x_1^2 x_2 - 25)$$

jos (x_1^*, x_2^*) on optimi, niin myös $(-x_1^*, x_2^*)$ on optimi

Geometrisesti selvää, että optimi on

$$\begin{cases} L_{x_1} = 0 \\ L_{x_2} = 0 \\ L_\lambda = 0 \end{cases} \iff \begin{cases} 2x_1 + 2\lambda x_1 x_2 = 0 \\ 2x_2 + \lambda x_1^2 = 0 \\ x_1^2 x_2 - 25 = 0 \end{cases}$$

$$\iff \begin{cases} 2x_1(1 + \lambda x_2) = 0 \\ \lambda = -2x_2/x_1^2 \\ x_1^2 x_2 = 25 \end{cases}$$ sijoitetaan λ yhtälöön #1

$$\iff \begin{cases} 1 - 2x_2^2/x_1^2 = 0 \\ x_1^2 x_2 = 25 \\ \lambda = -2x_2/x_1^2 \end{cases}$$ $\to x_1^2 = 2x_2^2$ sijoitetaan yhtälöön #2, #3

$$\iff \begin{cases} 2x_2^3 = 25 \\ x_1 = \sqrt{2}\, x_2 \\ \lambda = -1/x_2 \end{cases} \iff \begin{cases} x_2 = 2{,}32 \\ x_1 = 3{,}28 \\ \lambda = -0{,}43 \end{cases}$$

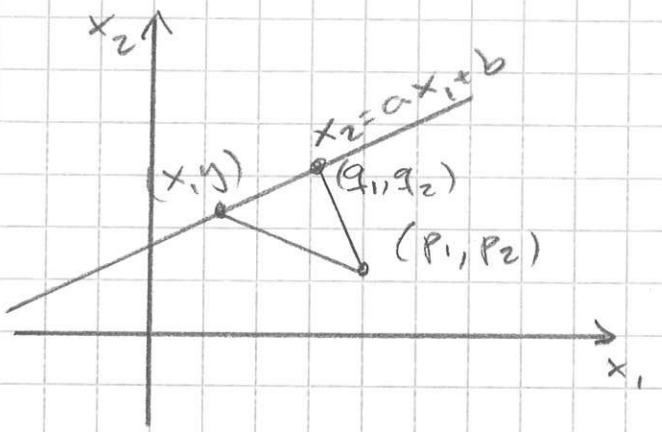
Vastaus

$$(x_1^*, x_2^*)^T = \begin{pmatrix} 3{,}28 \\ 2{,}32 \end{pmatrix}$$

tai

$$(x_1^*, x_2^*)^T = \begin{pmatrix} -3{,}28 \\ 2{,}32 \end{pmatrix}$$

2. On annettu tason suora $\ell = \{(x_1, x_2); x_2 = ax_1 + b\}$ ja tason piste $p = (p_1, p_2)$. Etsi käyttämällä Lagrangen kertoimien menetelmää suoran ℓ piste $q = (q_1, q_2)$, joka on lähinnä pistettä $p = (p_1, p_2)$.



$$f(x,y) = (x-p_1)^2 + (y-p_2)^2$$

$$\begin{cases} \min \; f(x,y) = (x-p_1)^2 + (y-p_2)^2 \\ \text{ehdolla} \quad y = ax + b \end{cases}$$

Lagrangen funktio

$$L = (x-p_1)^2 + (y-p_2)^2 + \lambda(y - ax - b)$$

$$\begin{cases} L_x = 0 \\ L_y = 0 \\ L_\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} 2(x-p_1) - a\lambda = 0 \\ 2(y-p_2) + \lambda = 0 \\ y - ax - b = 0 \end{cases} \to \lambda = -2(y-p_2)$$

$$\Leftrightarrow \begin{cases} (x-p_1) + a(y-p_2) = 0 \\ y - ax - b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + ay = p_1 + p_2 \\ -ax + y = b \end{cases} \left| \begin{array}{l} \cdot 1 \\ \cdot(-a) \end{array} \right| \begin{array}{l} \cdot a \\ \cdot 1 \end{array}$$

$$\Leftrightarrow \begin{cases} x = (p_1 + p_2 - ab)/(1+a^2) \\ y = (ap_1 + ap_2 + b)/(1+a^2) \end{cases}$$

$$\therefore \quad q_1 = \frac{(p_1+p_2) - ab}{1+a^2}, \quad q_2 = \frac{a(p_1+p_2)+b}{1+a^2}$$

Tarkistuksia

Tarkistetaan, onko piste (q_1, q_2) suoran $x_2 = ax_1 + b$ piste

$$aq_1 + b = a\left(\frac{p_1 + p_2 - ab}{1+a^2}\right) + b$$

$$= \frac{a(p_1+p_2) - a^2 b}{1+a^2} + \frac{b + a^2 b}{1+a^2}$$

$$= \frac{a(p_1+p_2) + b}{1+a^2} = q_2 \quad \text{ok}$$

vektorin $\vec{v} = \begin{pmatrix} 1 \\ a \end{pmatrix}$ (suoran suuntavektorin)
ja vektorin $\vec{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$

tulee olla kohtisuorassa.

$$\vec{v}^T \vec{PQ} = 1 \cdot (q_1 - p_1) + a(q_2 - p_2)$$

$$= \frac{(p_1+p_2) - ab}{1+a^2} - p_1 + a\left(\frac{a(p_1+p_2)+b}{1+a^2} - p_2\right)$$

$$= \frac{(p_1+p_2) - ab - p_1 - a^2 p_1 + a^2(p_1+p_2) + ab - p_2 - a^2 p_2}{1+a^2}$$

$$= 0 \quad \text{ok}$$

3. Tarkastelemme kolmen hyödykkeen Markowitzin ongelmaa: Sijoittaja haluaa maksimoida tuottonsa ja minimoida riskinsä. Hänen on päätettävä omaisuutensa osuudet $w = [w_1\ w_2\ w_3]^T$, jotka hän sijoittaa hyödykkeisiin 1, 2 ja 3. Luonnollisesti on oltava $w_1 + w_2 + w_3 = 1$. Hyödykkeet tuottavat $\mu = [\mu_1\ \mu_2\ \mu_3]^T = [01{,}1\ 0{,}04\ 0{,}02]^T$. ↑ po. 0,1
Kokonaistuotto salkulle w on siis

$$m(w) \;=\; \mu^T w.$$

Hyödykkeiden tuottojen kovarianssit ovat

$$K \;=\; \begin{bmatrix} 0{,}09 & -0{,}0072 & 0 \\ -0{,}0072 & 0{,}0144 & 0{,}0001 \\ 0 & 0{,}0001 & 0{,}01 \end{bmatrix},$$

jolloin salkun w riskin neliö on

$$s^2(w) \;=\; w^T K w.$$

Sijoittaja on valmis riskiin $s(w) = 20\%$. Kuinka hänen tulee jakaa varallisuutensa?

$$\begin{cases} \max\ m(w) = \mu^T w \\ \text{ehdoilla}\quad w_1 + w_2 + w_2 = 1 \\ \qquad\qquad w^T K w = 0{,}20^2 \end{cases}$$

$$\begin{cases} \max\ m(w) = 0{,}10 w_1 + 0{,}04 w_2 + 0{,}02 w_3 \\ \text{ehdoilla}\quad w_1 + w_2 + w_3 = 1 \\ \qquad 0{,}09 w_1^2 - 0{,}0144 w_1 w_2 + 0{,}0144 w_2^2 + 0{,}0002 w_2 w_3 + 0{,}01 w_3^2 = 0{,}04 \end{cases}$$

$$L = 0{,}10 w_1 + 0{,}04 w_2 + 0{,}02 w_3 + \lambda_1 (w_1 + w_2 + w_3 - 1)$$
$$+ \lambda_2 (0{,}09 w_1^2 - 0{,}0144 w_1 w_2 + 0{,}0144 w_2^2 + 0{,}0002 w_2 w_3 + 0{,}01 w_3^2 - 0{,}04)$$

$$\begin{cases} L_{w_1} = 0 \\ L_{w_2} = 0 \\ L_{w_3} = 0 \\ L_{\lambda_1} = 0 \\ L_{\lambda_2} = 0 \end{cases} \Leftrightarrow \begin{cases} 0{,}10 + \lambda_1 + \lambda_2 (0{,}18 w_1 - 0{,}0144 w_2) = 0 \\ 0{,}04 + \lambda_1 + \lambda_2 (-0{,}0144 w_1 + 0{,}0288 w_2 + 0{,}0002 w_3) = 0 \\ 0{,}02 + \lambda_1 + \lambda_2 (0{,}0002 w_2 + 0{,}02 w_3) = 0 \\ w_1 + w_2 + w_3 - 1 = 0 \\ 0{,}09 w_1^2 - 0{,}0144 w_1 w_2 + 0{,}0144 w_2^2 + 0{,}0002 w_2 w_3 + 0{,}01 w_3^2 - 0{,}04 = 0 \end{cases}$$

Melko hankala yhtälöryhmä

```
(%i2) g1: w1+w2+w3-1;
(%o2)                    w3 + w2 + w1 - 1
(%i3) g2: 0.09*w1^2-0.0144*w1*w2+0.0144*w2^2+0.0002*w2*w3+0.01*w3^2-0.04;
               2                                          2
(%o3) 0.01 w3   + 2.0000000000000001E-4 w2 w3 + 0.0144 w2   - 0.0144 w1 w2
                                                                       2
                                                            + 0.09 w1   - 0.04
(%i4) u1: 0.18*w1-0.0144*w2;
      u2: -0.0144*w1+0.0288*w2+0.0002*w3;
      u3: 0.0002*w2+0.02*w3;
(%o4)                     0.18 w1 - 0.0144 w2
(%o5)         2.0000000000000001E-4 w3 + 0.0288 w2 - 0.0144 w1
(%o6)               0.02 w3 + 2.0000000000000001E-4 w2
```



```
(%i7) solve([0.10+L1+L2*u1=0, 0.04+L1+L2*u2=0,0.02+L1+L2*u3=0, g1=0, g2=0], [w1,w2,w3,L1,L2]);
`rat' replaced 0.1 by 1/10 = 0.1
`rat' replaced 0.18 by 9/50 = 0.18
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.04 by 1/25 = 0.04
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.0288 by 18/625 = 0.0288
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.02 by 1/50 = 0.02
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.02 by 1/50 = 0.02
`rat' replaced -0.04 by -1/25 = -0.04
`rat' replaced 0.09 by 9/100 = 0.09
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.0144 by 9/625 = 0.0144
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.01 by 1/100 = 0.01
(%o7) [[w1 = - (656825 sqrt(244637007) + 1206355 sqrt(3) sqrt(37) sqrt(97)
 sqrt(22721) - 4961370744)/55606209912,
       2230 sqrt(37) sqrt(22721) - 96300 sqrt(3) sqrt(97)
w2 = - ---------------------------------------------------,
                  241271 sqrt(3) sqrt(97)
     9165 sqrt(37) sqrt(22721) + 246888 sqrt(3) sqrt(97)
w3 = ---------------------------------------------------,
                  482542 sqrt(3) sqrt(97)
       423679 sqrt(37) sqrt(22721) + 4976280 sqrt(3) sqrt(97)
L1 = - -------------------------------------------------------,
                  12063550 sqrt(37) sqrt(22721)
     40 sqrt(291)
L2 = ------------], [w1 = (656825 sqrt(244637007)
     sqrt(840677)
 + 1206355 sqrt(3) sqrt(37) sqrt(97) sqrt(22721) + 4961370744)/55606209912
```



```
(%i8) float(solve([0.10+L1+L2*u1=0, 0.04+L1+L2*u2=0,0.02+L1+L2*u3=0, g1=0, g2=0], [w1,w2,w3,L1,L2]));
`rat' replaced 0.1 by 1/10 = 0.1
`rat' replaced 0.18 by 9/50 = 0.18
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.04 by 1/25 = 0.04
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.0288 by 18/625 = 0.0288
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.02 by 1/50 = 0.02
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.02 by 1/50 = 0.02
`rat' replaced -0.04 by -1/25 = -0.04
`rat' replaced 0.09 by 9/100 = 0.09
`rat' replaced -0.0144 by -9/625 = -0.0144
`rat' replaced 0.0144 by 9/625 = 0.0144
`rat' replaced 2.0000000000000001E-4 by 1/5000 = 2.0000000000000001E-4
`rat' replaced 0.01 by 1/100 = 0.01
(%o8) [[w1 = - 0.434850656697, w2 = - 0.097648018888123,
w3 = 1.532498675585124, L1 = - 0.042795293785286, L2 = 0.7442038337351],
[w1 = 0.61329729968352, w2 = 0.89592050086897, w3 = - 0.50921780055248,
L1 = - 0.027445887293253, L2 = - 0.7442038337351]]
```

Syntynyt yhtälöryhmä oli hankala ratkaista. Siksi tutkimmekin vielä mikä on pienin mahdollinen riski, kun tuotto = X

Vastaus alkuperäiseen ongelmaan on

Suurin X, jolle $\min\limits_{\text{tuotto}=X} \text{riski} \leq 0{,}04$

$$\begin{cases} \min\ g(w) = w^T K w & \leftarrow \text{riski} \\ \text{ehdoilla}\quad w_1 + w_2 + w_3 = 1 & \\ \qquad \mu^T w = X & \leftarrow \text{tuotto} \end{cases}$$

$$\begin{cases} \min\ w^T K w = 0{,}09 w_1^2 - 0{,}0144 w_1 w_2 + 0{,}0144 w_2^2 + 0{,}0002 w_2 w_3 + 0{,}01 w_3^2 \\ \text{ehdoin}\quad w_1 + w_2 + w_3 = 1 \\ \qquad 0{,}10 w_1 + 0{,}04 w_2 + 0{,}02 w_3 = X \end{cases}$$

$$\begin{aligned} L = {} & 0{,}09 w_1^2 - 0{,}0144 w_1 w_2 + 0{,}0144 w_2^2 + 0{,}0002 w_2 w_3 + 0{,}01 w_3^2 \\ & + \lambda_1 (w_1 + w_2 + w_3 - 1) \\ & + \lambda_2 (0{,}10 w_1 + 0{,}04 w_2 + 0{,}02 w_3 - X) \end{aligned}$$

$$\begin{cases} L_{w_1} = 0 \\ L_{w_2} = 0 \\ L_{w_3} = 0 \\ L_{\lambda_1} = 0 \\ L_{\lambda_2} = 0 \end{cases} \Leftrightarrow \begin{cases} 0{,}18 w_1 - 0{,}0144 w_2 + \lambda_1 + 0{,}10 \lambda_2 = 0 \\ -0{,}0144 w_1 + 0{,}0288 w_2 + 0{,}0002 w_3 + \lambda_1 + 0{,}04 \lambda_2 = 0 \\ 0{,}0002 w_2 + 0{,}02 w_3 + \lambda_1 + 0{,}02 \lambda_2 = 0 \\ w_1 + w_2 + w_3 = 1 \\ 0{,}10 w_1 + 0{,}04 w_2 + 0{,}02 w_3 = X \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 0{,}18 & -0{,}0144 & 0 & 1 & 0{,}10 \\ -0{,}0144 & 0{,}0288 & 0{,}0002 & 1 & 0{,}04 \\ 0 & 0{,}0002 & 0{,}02 & 1 & 0{,}02 \\ 1 & 1 & 1 & 0 & 0 \\ 0{,}10 & 0{,}04 & 0{,}02 & 0 & 0 \end{bmatrix}}_{= A} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ X \end{bmatrix}$$

$$\Leftrightarrow \quad A\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}^* = A^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \end{bmatrix} = \begin{bmatrix} -0{,}265679 + 10{,}105241x \\ 0{,}062715 + 9{,}579038x \\ 1{,}202964 - 19{,}684278x \\ -0{,}042271 + 0{,}909963x \\ 0{,}909963 - 25{,}909686x \end{bmatrix}$$

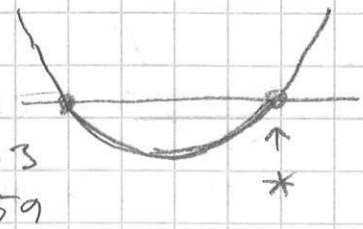
pienin riski tuottotasolla x on

$$g(x) = w^{*T} K w^*$$
$$= \ldots$$
$$= 12{,}954843x^2 - 0{,}909963x + 0{,}0211355$$

maxima

$g(x) = 0{,}04 \Leftrightarrow x = -0{,}01674$ tai $x = 0{,}0869822$

Valitaan $x = 0{,}0869822$



$w_1 = -0{,}265679 + 10{,}105241 \cdot 0{,}0869822 \approx 0{,}6133$
$w_2 = 0{,}062715 + 9{,}579038 \cdot 0{,}0869822 \approx 0{,}8959$
$w_3 = 1{,}202964 - 19{,}684278 \cdot 0{,}0869822 \approx -0{,}5092$

Vastaus max tuotto on 8,70 %, kun
$w_1 \approx 0{,}613$, $w_2 \approx 0{,}896$, $w_3 \approx -0{,}509$

(Hyödykettä 3 myytiin lyhyeksi)

```
(%i2) A: matrix(
       [0.18,-0.0144,0,1,0.10],
       [-0.0144,0.0288,0.0002,1,0.04],
       [0,0.0002,0.02,1,0.02],
       [1,1,1,0,0],
       [0.10,0.04,0.02,0,0]
      );
```

1

$$(\%o2) \quad \begin{bmatrix} 0.18 & -0.0144 & 0 & 1 & 0.1 \\ -0.0144 & 0.0288 & 2.0000000000000001E-4 & 1 & 0.04 \\ 0 & 2.0000000000000001E-4 & 0.02 & 1 & 0.02 \\ 1 & 1 & 1 & 0 & 0 \\ 0.1 & 0.04 & 0.02 & 0 & 0 \end{bmatrix}$$

```
(%i3) C: matrix(
       [1,0,0,0,0],
       [0,1,0,0,0],
       [0,0,1,0,0]
      );
```

$$(\%o3) \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$


```
(%i4) rhs: matrix(
       [0],
       [0],
       [0],
       [1],
       [x]
      );
```

2

$$(\%o4) \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \end{bmatrix}$$

```
(%i5) wl: invert(A).rhs;
```

$$(\%o5) \quad \begin{bmatrix} 10.10524054982818\ x - 0.26567869415808 \\ 9.579037800687283\ x + 0.062714776632302 \\ 1.202963917525773 - 19.68427835051547\ x \\ 0.90996348797251\ x - 0.042271091065292 \\ 0.90996348797251 - 25.90968642611684\ x \end{bmatrix}$$

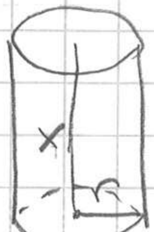
3

```
(%i6) w: C.wl;
                  [ 10.10524054982818 x - 0.26567869415808  ]
                  [                                         ]
(%o6)             [ 9.579037800687283 x + 0.062714776632302 ]
                  [                                         ]
                  [ 1.202963917525773 - 19.68427835051547 x ]
(%i7) K: matrix(
       [0.09,-0.0072,0],
       [-0.0072,0.0144,0.0001],
       [0,0.0001,0.01]
      );
                        [   0.09     - 0.0072     0    ]
                        [                              ]
(%o7)                   [ - 0.0072   0.0144    1.0E-4  ]
                        [                              ]
                        [    0       1.0E-4     0.01   ]
(%i8) g: expand(transpose(w).K.w);
                                   2
(%o8)    12.95484321305842 x  - 0.90996348797251 x + 0.021135545532646
(%i9) float(solve([g=0.04], [x]));
`rat' replaced -0.01886445446735 by -718/38061 = -0.01886445442842
`rat' replaced -0.9099634879725 by -6731/7397 = -0.9099634987157
`rat' replaced 12.95484321305842 by 17787/1373 = 12.95484340859432
(%o9)          [x = - 0.016741012709916, x = 0.086982193557538]
```


4

```
(%i10) w;
                  [ 10.10524054982818 x - 0.26567869415808  ]
                  [                                         ]
(%o10)            [ 9.579037800687283 x + 0.062714776632302 ]
                  [                                         ]
                  [ 1.202963917525773 - 19.68427835051547 x ]
(%i11) subst(0.086982193557538, x, w);
                           [  0.61329729529256  ]
                           [                    ]
(%o11)                     [  0.89592049670666  ]
                           [                    ]
                           [ - 0.50921779199922 ]
```

4. Kaljatölkki on sylinterin muotoinen ja sen on oltava tilavuudeltaan 0,5 litraa. Purkittamo haluaa minimoida tölkkiin käytettävän materiaalin. Kuinka korkea on tölkin oltava?



Tilavuus $\pi r^2 x$

pinta-ala $2\pi r x + 2\pi r^2$

Merkitään $V = 0{,}5\,l$

$$\begin{cases} \text{Min } & 2\pi r x + 2\pi r^2 \\ \text{ehdolla } & \pi r^2 x = V \end{cases}$$

$$L = 2\pi r x + 2\pi r^2 + \lambda(\pi r^2 x - V)$$

$$\begin{cases} L_x = 0 \\ L_r = 0 \\ L_\lambda = 0 \end{cases} \Leftrightarrow \begin{cases} 2\pi r + \lambda \pi r^2 = 0 \\ 2\pi x + 4\pi r + 2\lambda\pi r x = 0 \\ \pi r^2 x - V = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2 + \lambda r = 0 \quad \rightarrow \lambda r = -2 \\ x + 2r + \lambda r x = 0 \\ \pi r^2 x = V \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda r = -2 \\ 2r = x \\ \pi r^2 x = V \end{cases} \quad r = \frac{x}{2}, \quad V = \pi \cdot \left(\frac{x}{2}\right)^2 \cdot x$$

$$\therefore \quad V = \frac{\pi}{4} \cdot x^3 \quad \rightarrow \quad x = \sqrt[3]{\frac{4V}{\pi}} = \sqrt[3]{\frac{4 \cdot 500\,cm^3}{\pi}}$$

$$x \approx 8{,}60\,cm$$
$$r \approx 4{,}30\,cm$$

halkaisija (2r) = korkeus (x) !?!