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Power series expansions for fractional Brownian Motions [1]

Fractional Brownian Motions (FBM) are selfsimilar Gaussian processes with hölder-continuous paths, that can be represented as fractional integrals ($H > \frac{1}{2}$) or derivatives ($H < \frac{1}{2}$) of Brownian Motions (BM), allow an stochastic calculus, have long-range memory and are of interest in finance and network traffic. For the theory and for the numerical simulation of FBM series expansions are of interest. The authors present one approach for series expansions which bases on series expansions of BM.

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1. Introduction

The **Fractional Brownian Motion (FBM)** is a Gaussian process $Z_H = (Z_H(t), t \geq 0)$ whose covariance function is

$$R_H(t,s) := \mathbb{E}[Z_H(t)Z_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - \mid t-s \mid^{2H}).$$

Here $H \in (0,1)$ is the **Hurst index**. If $H = \frac{1}{2}$ then FBM is the BM.

The FBM has some characteristic properties. First, it is H**-selfsimilar**, i.e. for any $\alpha > 0$

$$(Z(\alpha t), t \geq 0) \stackrel{d}{=} \left(\alpha^H Z(t), t \geq 0\right).$$

Second, it admits various **integral representations**. The **Mandelbrot–Van-Ness-representation** (cf. [7]) shows that the FBM is fractionally

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integrated ($H \geq \frac{1}{2}$) or differentiated ($H \leq \frac{1}{2}$) BM. Another integral representation is (cf. [8])

$$\begin{aligned} Z_H(t) &= \int_0^\infty z_H(t,s)dZ_{1/2}(s) \qquad t \in [0,\infty) \\ z_H(t,s) &= c_H\, s^{H-\frac{1}{2}} \tfrac{d}{ds} \int_s^t u^{H-\frac{1}{2}}\, (u-s)^{H-\frac{1}{2}}\, du, \quad t,s \in [0,\infty), \\ c_H &= \sqrt{\tfrac{(2H+\frac{1}{2})\,\Gamma(\frac{1}{2}-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)}} \end{aligned} \tag{1}$$

Here z_H is a square integrable Volterra kernel with singularities at 0 for $H \in (0,1)$ and also at t for $H < \frac{1}{2}$. Third, from the Kolmogorov-Čentsov theorem one concludes that the FBM admits a version with **Hölder-continuous** trajectories of order $\gamma < H$. Fourth, it is possible to define **integrals with respect to the FBM**. As the FBM is **not a semimartingale** the common Itô-calculus does not apply. But a FBM has sufficient properties which allow to define meaningful integrals (see e.g. [2] and [3] for different approaches).

The mentioned properties make the FBM interesting for **applications**, e.g. in the modeling of network traffic and financial time series. Series expansions of the FBM are of interest for two reasons. They provide a basis for simulation methods of an FBM and are a useful theoretical tool.

2. Series expansions

2.1. Reproducing Kernel Hilbert Spaces (RKHS) and series expansions of Gaussian processes

We shall consider series expansions on a compact interval. Since the FBM is selfsimilar we may (and shall) as well take that interval to be $[0,1]$.

Let us start with some generalities of series expansions of Gaussian processes. Let $X = (X(t), t \in [0,1])$ be a centered Gaussian process with covariance function K. Its **linear space** $\mathcal{H}^1_X$ is the closure in $\mathcal{L}^2(\Omega,\mathbb{P})$ of

$$\operatorname{span}\{X(t) \mid t \in [0,1]\}.$$

If $\mathcal{H}^1_X$ is separable then we have an expansion of $X(t)$ in $\mathcal{L}^2(\Omega,\mathbb{P})$, viz.

$$X(t) = \sum_{n=0}^\infty \mathbb{E}\left[X(t)Y_n\right] Y_n. \tag{2}$$

Here the Y_n's form a CONS of $\mathcal{H}^1_X$, i.e. they are independent standard Gaussian random variables. The independence of the summands yields that (2) also converges almost surely for all $t \in [0,1]$.

So, to construct an $\mathcal{L}^2(\Omega, \mathbb{P})$ and almost sure expansion of a separable Gaussian process we only need to find the functions $\mathbb{E}\left[X(\_)Y_n\right]$. To do this we introduce a space isometric to $\mathcal{H}^1_X$ by linearly expanding the relation

$$\Theta : X(t) \mapsto K(t, \_).$$

More precisely, set

$$\mathcal{S}_K := \operatorname{span}\{\, K(t, \_) \mid t \in [0,1]\,\}$$

and define an inner product on $\mathcal{S}_K$ by expanding the relation

$$\langle K(t, \_), K(s, \_)\rangle_K := K(t,s).$$

The **reproducing kernel Hilbert space (RKHS)** of X with covariance K, denoted by $\mathcal{H}_K$, is the closure of $\mathcal{S}_R$ with respect to $\langle\ ,\ \rangle_R$.

Now Θ is an isometry from $\mathcal{H}^1_X$ onto $\mathcal{H}_K$. If K is continuous then $\mathcal{H}_K$ (and hence $\mathcal{H}^1_X$) is separable. Thus, there exists an expansions (2) of X.

The inner product $\langle\ ,\ \rangle_K$ has a **reproducing property**: for $f \in \mathcal{H}_K$

$$f(t) = \langle f, K(t, \_)\rangle_K.$$

So, if $\{\,\psi_k\,\}_{k\in\mathbb{N}}$ is a CONS in $\mathcal{H}_K$ then the reproducing property yields

$$K(t, \_) = \sum_{k=0}^{\infty} \langle K(t, \_)\psi_k\rangle_K \psi_k = \sum_{k=0}^{\infty} \psi_k(t)\psi_k. \tag{3}$$

Thus, the application of the isometry Θ together with (3) yields that in $\mathcal{L}^2(\Omega, \mathbb{P})$ and almost surely for all $t \in [0,1]$ we have

$$X(t) = \Theta\left(K(t, \_)\right) = \Theta\left(\sum_{k=0}^{\infty} \psi_k(t)\psi_k\right) = \sum_{k=0}^{\infty} \psi_k(t) Y_k, \tag{4}$$

where $\{Y_k\}_{k\in\mathbb{N}} = \{\Theta(\psi_k)\}_{k\in\mathbb{N}}$ is a CONS in $\mathcal{H}^1_X$, i.e. they are independent standard Gaussian random variables. So, we have found a series expansion of the Gaussian process X if we know a CONS in $\mathcal{H}_K$. Moreover, by Itô–Nisio theorem (cf. [1], Theorem 3.8) the series (4) converge also almost surely uniformly on $[0,1]$ if and only if X has continuous paths.

2.2. Series expansions of a Fractional Brownian Motion

Using the kernel z_H the covariance of the FBM may be written as

$$R_H(t,s) = \int_0^{t\wedge s} z_H(t,x) z_H(s,x)\, dx.$$

Let us define an operator acting on $\mathcal{L}^2([0,1])$ by expanding the relation

$$\Psi_H : z_H(t, \_) \mapsto R_H(t, \_)$$

i.e.

$$(\Psi_H f)(t) = \int_0^t z_H(t,s)f(s)\,ds.$$

Now Ψ_H is injective (cf. [9]). Thus, it is an isometry from $\mathcal{L}^2([0,1])$ onto $\mathcal{H}_R$. So, combining (4) with Ψ_H and the continuity of the FBM we have:

Theorem 2.1. *The FBM admits the representation*

$$Z_H(t) = \sum_{k=0}^{\infty} \int_0^t z_H(t,s)\psi_k'(s)\,ds \cdot Y_k, \tag{5}$$

where $\{\psi_k'\}_{k\in\mathbb{N}}$ *is any CONS in* $\mathcal{L}^2([0,1])$ *and* $\{Y_k\}_{k\in\mathbb{N}} = \{(\Theta\circ\Psi_H)(\psi_k')\}_{k\in\mathbb{N}}$ *are independent standard Gaussian random variables. The series (5) converge in* $\mathcal{L}^2(\Omega,\mathbb{P})$ *and almost surely uniformly on* $[0,1]$.

Remark. Heuristically one obtains (5) as follows: the RKHS of the BM is $\{f = \int_0^{(\cdot)} f'(t)\,dt \,:\, f' \in \mathcal{L}^2([0,1])\}$. Then (1) yields

$$\begin{aligned} Z_H(t) &= \int_0^t z_H(t,s)\,dZ_{1/2}(s) \\ &= \int_0^t z_H(t,s)\,d\left[\sum_{k=0}^{\infty}\psi_k(s)\cdot Y_k\right] \\ &= \sum_{k=0}^{\infty}\int_0^t z_H(t,s)\psi_k'(s)\,ds\cdot Y_k. \end{aligned}$$

An unfortune fact is that for a given function in $\mathcal{L}^2([0,1])$ it is not easy to calculate the integral in (5). Nevertheless, we have the following.

Lemma 2.2. *Let* $\beta > H - \frac{1}{2}$ *and denote by* B *the Beta function. Then*

$$\int_0^t z_H(t,s)s^\beta\,ds \;=\; c_{H,\beta}t^{\beta+H+\frac{1}{2}} \tag{6}$$

where

$$c_{H,\beta} = \frac{\frac{1}{2}-H}{\beta+H+\frac{1}{2}}\mathrm{B}(H+\frac{1}{2},\beta-H+\frac{1}{2}).$$

Proof. Integrating by parts, changing the order of integration and setting $u = s/x$ we obtain

$$\begin{aligned} \int_0^t z_H(t,s)s^\beta\,ds &= -c_H\int_0^t s^{\frac{1}{2}-H}\frac{d}{ds}\int_s^t x^{H-\frac{1}{2}}(x-s)^{H-\frac{1}{2}}\,dx\,s^\beta\,ds \\ &= (\frac{1}{2}-H)c_H\int_0^t\int_s^t x^{H-\frac{1}{2}}(x-s)^{H-\frac{1}{2}}\,dx\,s^{\beta-H-\frac{1}{2}}\,ds \\ &= (\frac{1}{2}-H)c_H\int_0^t\int_0^x (x-s)^{H-\frac{1}{2}}s^{\beta-H-\frac{1}{2}}\,ds\,x^{H-\frac{1}{2}}\,dx \\ &= (\frac{1}{2}-H)c_H\int_0^t\int_0^1(1-u)^{H-\frac{1}{2}}u^{\beta-H-\frac{1}{2}}\,du\,x^{\beta+H-\frac{1}{2}}\,dx \end{aligned}$$

$$= (\frac{1}{2} - H)c_H \int_0^t \mathrm{B}(H + \frac{1}{2}, \beta - H + \frac{1}{2}) x^{\beta + H - \frac{1}{2}} \, dx.$$

The claim follows. □

By using the lemma above we obtain concrete expansions of the FBM by using polynomial expansions (or bases) in $\mathcal{L}^2([0,1])$. We give two examples.
Polynomial representation: The shifted Legendre polynomials

$$\tilde{\psi}_k^{\mathrm{poly}}(t) = \sum_{\ell=0}^{k} \left\{ \sum_{j=0}^{\lfloor (k-\ell)/2 \rfloor} d_{k,\ell,j} \right\} t^\ell$$

where

$$d_{k,\ell,j} = \frac{(-1)^{k-j-\ell}}{2^{k-\ell}} \binom{k}{j} \binom{2k-2j}{k} \binom{k-2j}{\ell}$$

form a CONS of $\mathcal{L}^2([0,1])$. So, we obtain a concrete series expansion

$$Z_H(t) = \sum_{k=0}^{\infty} \psi_k^{\mathrm{poly}}(t) \cdot Y_k \tag{7}$$

with

$$\psi_k^{\mathrm{poly}}(t) = t^{H+\frac{1}{2}} \sum_{\ell=0}^{k} c_{H,\ell} \left\{ \sum_{j=0}^{\lfloor (k-\ell)/2 \rfloor} d_{k,\ell,j} \right\} t^\ell.$$

Let us note that the expansion (7) is not suitable for simulation as it is computationally unstable.

Trigonometric representation: Expanding the cosine function

$$\tilde{\psi}_k^{\mathrm{trig}}(t) = \sqrt{2}\cos(k\pi t) = \sqrt{2} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(k\pi t)^{2\ell}}{(2\ell)!}$$

we obtain the expansion

$$Z_H(t) = \sum_{k=0}^{\infty} \psi_k^{\mathrm{trig}}(t) \cdot Y_k \tag{8}$$

with

$$\psi_k^{\mathrm{trig}}(t) = \frac{c_H \Gamma(\frac{1}{2} - H)}{H + \frac{1}{2}} t^{H+\frac{1}{2}} F_H\left(-\frac{1}{4}(k\pi t)^2 \right)$$

where F_H is the Hypergeometric function (see [6] for definition)

$$F_H(z) = {}_3F_4\left(\frac{5-2H}{4}, \frac{3+2H}{4}, \frac{3-2H}{4}; 1, \frac{5+2H}{4}, \frac{1}{2}, \frac{1}{2}\,;\, z \right).$$

Using asymptotic expansions of ${}_3F_4$ (cf. [6], p. 199) one obtains that

$$\psi_k^{\mathrm{trig}}(t) = b(H,t,k)k^{-(H+\frac{1}{2})}$$

where $b(H,t,k)$ is bounded. Note the analogue to the BM case, when

$$\psi_k^{\mathrm{trig}}(t) = \sqrt{2}\sin(k\pi t)(k\pi)^{-1}.$$

The fact that the rate in (8) is better with larger H was expected. Indeed, the paths of the FBM grow more erratic as H decreases.

Remark. By Kühn and Linde [5] the optimal rate of convergence in series expansions like (4) of the FBM is

$$\mathbb{E}\left[\sup_{t\in[0,1]}\left|\sum_{k=N+1}^{\infty}\psi_k(t)Y_k\right|\right] \sim N^{-H}\sqrt{\log N}.$$

There is an expansion due to Dzhaparidze and van Zanten [4] that obtains this optimal rate. It involves calculations of zeros of certain Bessel functions. In our case the convergence rate of course depends on the particular CONS used. Unfortunately, we do not know if e.g. the polynomial expansion (7) or the trigonometric expansion (8) is optimal in this sense.

3. Numerical aspects of approximations of FBM

Due to the applications mentioned in the introduction it is necessary to consider the **numerical simulation** of FBM. The direct way to simulate a FBM is to exploit its Gaussianity and to **compute it exactly**, i.e.:

$$\begin{pmatrix} Z_H(t_0) \\ \dots \\ Z_H(t_n) \end{pmatrix} = L_H(t_0,\dots,t_n)\begin{pmatrix} \epsilon(t_0) \\ \dots \\ \epsilon(t_n) \end{pmatrix}, \qquad n\in\mathbb{N} \tag{9}$$

where

$$R_H(t_0,\dots,t_n) = L_H(t_0,\dots,t_n)\,(L_H(t_0,\dots,t_n))^\top$$
$$R_H(t_0,\dots,t_n) = (\ R_H(t_i,t_j)\ )_{i,j\in\mathbb{N}_n} = (\ t_i^{2H}+t_j^{2H} - \mid t_i^{2H}-t_j^{2H}\mid\ )_{i,j=0,\dots n}$$
$$L_H(t_0,\dots,t_n) = (\ L_H(t_i,t_j)\ )_{i,j\in\mathbb{N}_n}$$
$$(\epsilon(t_0),\dots,\epsilon(t_n))^\top \sim N((0,\dots,0)^\top, diag(t_0,t_1-t_0,\dots,t_n-t_{n-1})).$$

This approach has two disadvantages:
1) The required $LL^\top$-decomposition of R_H is computationally expensive.
2) The computation of $Z_H(\tilde{t}_i)$ at a new lattice point $\tilde{t}_i\in(t_{i-1},t_i)$ in a grid $\{t_i\}_{i\in\mathbb{N}_n}$ of already computed path values $\{Z_H(t_i)\}_{i\in\mathbb{N}_n}$ can be computationally expensive. It requires appropriate $LL^\top$-decomposition, too.

One way to overcome the resource problems is to work with **truncated covariances**. In this approach one takes the covariance with a fixed number k of points and hence a restricted time-dependence into consideration.

$$\tilde{Z}_H(t_j) = \sum_{i=j-k\vee 0}^{j} L_H(t_j, t_l)\epsilon(t_l). \tag{10}$$

Introducing the mean values

$$\bar{m}(Z_H(t_i), \dots, Z_H(t_j)) = \sqrt{\mathbb{E}[|\ \frac{1}{j+1-i}\sum_{k=i}^{j} Z_H(t_k)\ |^2]}, \tag{11}$$

and the denotation

$$L_H(i_1, i_2; j_1, j_2)) = (\ L_H(t_i, t_j)\ )_{i=i_1,\dots,i_2; j=j_1,\dots,j_2}$$
$$R_H(i_1, i_2; j_1, j_2)) = (\ R_H(t_i, t_j)\ )_{i=i_1,\dots,i_2; j=j_1,\dots,j_2}$$

one can compute the mean quotient:

$$\frac{\bar{m}(\tilde{Z}_H(t_{n-k}),\dots,\tilde{Z}_H(t_n))}{\bar{m}(Z_H(t_{n-k}),\dots,Z_H(t_n))}$$
$$= \sqrt{\frac{1^\top R_H(n-k+1,n;n-k+1,n)1 - 1^\top R_H(n-k+1,n;1,n-k)R_H(1,n-k;1,n-k)^{-1}R_H(n-k+1,n;1,n-k)^\top 1}{1^\top R_H(n-k+1,n;1,n-k)1}}. \tag{12}$$

Considering the special case of $H = 0.5$, $t_i = ih$, $h > 0$, $i \in \mathbb{N}_n$, straightforward calculations show that

$$\frac{\bar{m}(\tilde{Z}_{0.5}(t_{n-k}),\dots,\tilde{Z}_{0.5}(t_n))}{\bar{m}(Z_{0.5}(t_{n-k}),\dots,Z_{0.5}(t_n))} = \sqrt{\frac{k+2}{3n-2k+2}} = \mathcal{O}(\frac{1}{\sqrt{n}}) \underset{n\to\infty}{\longrightarrow} 0\ . \tag{13}$$

This clearly indicates a lack in accuracy for growing n. Hence this approach seems not to be suitable for a simulation of FBM at many time values.

Truncated series approximations of FBM. The truncation of series representations of FBM is an approximation method which is an alternative to the exact simulation and to other approximation methods. Truncated series (TSR) have the following advantage:
1. TSR do not use time step discretizations of the process.
2. A FBM can be approximated at new lattice points.
However, the benefit of this approximation method depends crucially on the right truncation point. In this part we want to treat some questions which arise in this context.

The next definitions are required for the coming parts.

Definition. Let $Z(t) = \sum_{k=0}^{\infty} \psi_k(t)\, Y_k$ be a series expansion of Z, $Y_k \sim N(0,1)$ i.i.d., $N \in \mathbb{N}$, $p \in [1,\infty]$. We call:

$\hat{Z}_H^N := \sum_{k=0}^{N} \psi_k\, Y_k$ **Approximation** of Z at the **truncation point** N,
$\Delta\hat{Z}_H^N := Z_H^N - \hat{Z}_H^N$ **Truncation error** of $\hat{Z}_H^N$.
$Err_{p,u}(\Delta\hat{Z}_H^N) := \mathbb{E}[\ sup\{\ |\Delta\hat{Z}_H^N|^p\ \}\ ]$ **uniform approximation error**,
$Err_p(\Delta\hat{Z}_H^N) := \mathbb{E}[\ \|\Delta\hat{Z}_H^N\|_{L^p([0,1])}\ ]$ $\mathcal{L}^p$**-approximation error**.

Next we want to extend slightly our approach of constructing series expansions to restrict not only to ONS in $\mathcal{L}^2([0,1])$ but to include complete systems as well. Let $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$ be an arbitrary complete system in $\mathcal{L}^2([0,1])$. Denote $\vec{\varphi} = (\varphi_0, \ldots, \varphi_N, \ldots)^\top$, $C(\vec{\varphi}) = (\langle \varphi_i, \varphi_j \rangle)_{i,j\in\mathbb{N}}$ and $L(\vec{\varphi})$ the Cholesky factor of $C(\vec{\varphi})$. Then from section 2 it follows that $\vec{\tilde{\varphi}} = L(\vec{\varphi})^{-1}\vec{\varphi}$ is an ONS in $\mathcal{L}^2([0,1])$ and

$$\sum_{k=0}^{\infty} \psi_k(t) Y_k$$

is a series expansion for a FBM where

$$\begin{aligned}
\vec{\psi}(t) &= (\psi_0(t), \ldots, \psi_N(t), \ldots)^\top \quad = \int_0^t z_H(t,s)\, L(\varphi)^{-1} \vec{\varphi}(s)\, ds \\
&= L(\vec{\varphi})^{-1} \int_0^t z_H(t,s)\, \vec{\varphi}(s)\, ds = L(\vec{\varphi})^{-1}\, \vec{\tilde{\psi}}(t) \\
\vec{\tilde{\psi}}(t) &= \left(\int_0^t z_H(t,s)\, \varphi_0(s)\, ds, \ldots, \int_0^t z_H(t,s)\, \varphi_N(s)\, ds, \ldots \right)^\top.
\end{aligned}$$

Now we can see that the usage of truncated series depends on three factors:

1. The computational costs for the orthonormalization.
   These are fixed initial costs of this method.
2. The computational costs for the FBM-coefficient functions.
   This are running costs of this method.
3. The determination of the truncation point.
   This affects the error control.

In the rest of this section we deal with these questions in order to determine how to choose best an appropriate complete system for numerical purposes. The question 1 and 2 are related. This is because the choice of a complete systems is a degree of freedom which can be used in order to make the computations of $\vec{\tilde{\psi}}(t)$ as easy as possible. On the other hand, a complete system in $\mathcal{L}^2([0,1])$ is not necessarily an ONS. Hence the initial orthonormalization affects the numerical method. The next two examples illustrate the tradeoff between question 1 and 2.

Example. Consider the complete system $\varphi = \{t^k\}_{k\in\mathbb{N}}$. Then $\tilde{\psi}_k(t) = c_{H,k} t^{k+H+\frac{1}{2}}$ can be computed easily. It holds $C(\varphi) = (\frac{1}{i+j+1})_{i,j\in\mathbb{N}}$. It can be observed that the $LL^\top$-factorization of $C(\varphi)_N$ is numerically unstable: due to rounding errors the algorithm detects negative definite matrix. With an algorithm basing on Gram-Schmidt orthonormalization the problem can be overcome. But the structural problems remain as the elements $l_{i,j}^{-1}$, $i,j \in \mathbb{N}$, of $L(\varphi)^{-1}$ have alternating signs and as $\min_{i,j\in\mathbb{N}} \{ \, |l_{ij}^{-1}| \mid l_{ij}^{-1} \neq 0 \, \} \ll \max_{i,j\in\mathbb{N}} \{ \, |l_{ij}^{-1}| \, \}$.

Example. Consider the φ^{trig}. Being an ONS φ^{trig} does not require an orthonormalization. However, $\tilde{\psi}_k(t) = \frac{c_H\, \Gamma(\frac{1}{2}-H)}{H+\frac{1}{2}} t^{H+\frac{1}{2}} F_H(-\frac{1}{4}(k\pi t)^2)$, and so far no satisfying method for an efficient computation of F_H has been found.

From the preceding two examples we have seen that it is hard to find complete systems having low orthonormalization costs and ψ which are easy to compute. Hence a hybrid method seems to be recommendable: 1) to determine a truncation point N such that a given error bound $\frac{\epsilon}{2}$ is guaranteed, 2) to choose a ONS φ which can be represented as power series, 3) to truncate the power series of first N basis functions to $\hat{\varphi}_k$, $k \in \mathbb{N}$, such that the error by using $\hat{\varphi}_k$ instead of the exact basis functions φ_k, $k \in \mathbb{N}$, does not exceed $\frac{\epsilon}{2}$, 4) to use the fast computation for the approximation as the z_H-transforms of $\{t^k\}_{k\in\mathbb{N}}$ are known.

The dependence on the truncation points leads us to the question 3 which we treat in the rest of this section. One would like to determine the truncation point such that a given error bound $\epsilon \in (0,\infty)$ is guaranteed. The desirable error measure would be $Err_{2,u}$. But as this is often hard to achieve Err_2 is considered to be sufficient for practical purposes. That is it is necessary to find estimates for

$$\mathbb{E}[\, \|\Delta \hat{Z}_H^N\|_{\mathcal{L}^2([0,1])}\,] \;=\; \sum_{k=N}^{\infty} \|\psi_k\|^2_{\mathcal{L}^2([0,1])} \;=\; \int_0^1 \vec{\tilde{\psi}}^{\top}_{N,\infty}(t) C(\vec{\varphi})^{-1}_{N,\infty} \vec{\tilde{\psi}}_{N,\infty}(t)\, ds$$

where

$$\vec{\tilde{\psi}}^{\top}_{N,\infty} = (\tilde{\psi}_N, \ldots)^{\top},\; C(\vec{\varphi})_{N,\infty} = (C(\vec{\varphi})_{i,j})_{i,j=N,\ldots}.$$

The above error representation shows that the orthonormalization has an impact on the estimation of the truncation error too. This is either due to the matrix $C(\vec{\varphi})^{-1}_{N,\infty}$ in the second equality or due to $\psi = L(\varphi)^{-1}_{\infty}\tilde{\psi}$ in the first equality. So for the determination of a truncation point explicit knowledge about the possibly ill-conditioned matrix $C(\vec{\varphi})$ is necessary.

Remark. For the complete system $\{t^k\}_{k\in\mathbb{N}}$ used as test case this did not lead to satisfying estimates of the truncation error.

If φ is already an ONS then $C(\vec{\varphi}) = I_\infty$. In this case one can use the following inequality

$$\|\psi_k\|^2_{\mathcal{L}^2([0,1])} \le \sup_{t\in[0,1]} \{\, | \int_0^1 z_H(t,s)\,\varphi_k(s)\,ds\,|\,\}^2$$

for the determination of the truncation point by finding good upper estimates for $|\int_0^1 z_H(t,s)\,\varphi_k(s)\,ds\,|$.

Remark. For φ^{trig} this approach lead to a the convergence rate $N^{-(H+\frac{1}{2})}$ in terms of Err_2 but not to satisfying results concerning an error estimation.

For cases where it is hard to estimate explicitely $|\int_0^1 z_H(t,s)\,\varphi_k(s)\,ds\,|$ a different approach relying on partial integration can be chosen for $H > \frac{1}{2}$. This is what we present now. In the following the essential proofs are deferred to the end of this section. For technical convenience we introduce the following functions

Definition. $G_H(u) := \int_u^1 (1-v)^{H-\frac{3}{2}} v^{-2H} dv \qquad u \in [0,1],$

$$\overline{G}_H(u) := \begin{cases} \frac{1}{2(H-\frac{1}{2})} & u = 0 \ , \\ \frac{\int_u^1 (1-v)^{H-\frac{3}{2}} v^{-2H} dv}{(1-u)^{H-\frac{1}{2}}\, u^{-2(H-\frac{1}{2})}} & u \in (0,1) \ , \\ \frac{1}{H-\frac{1}{2}} & u = 1 \ . \end{cases}$$

Lemma 3.1. It holds:

(i) $G_H \in \mathcal{C}((0,1], \mathbb{R}_+) \cap \mathcal{C}^\infty((0,1), \mathbb{R}_+)$,

(ii) G_H has a pole of order $2(H - \frac{1}{2})$ at 0 ,

(iii) $\lim\limits_{u \uparrow 1} \frac{G_H(u)}{(1-u)^{H-\frac{1}{2}}} = \frac{1}{H-\frac{1}{2}}, \qquad G_H(u) = \mathcal{O}((1-u)^{H-\frac{1}{2}})$.

Furthermore, we shall make use of fractional derivatives and integrals.

Definition. Let $f \in \mathcal{L}([0,1]^2, \mathbb{R}), \alpha \in (0,1), t, s \in [0,1]$.

(i) $D^\alpha_\pm(s) f(t,s) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_0^1 f(t,u)\,(s-u)^{-\alpha}_\mp du,$

(ii) $I^\alpha_\pm(s) f(t,s) = \frac{1}{\Gamma(\alpha)} \int_0^1 f(t,u) (s-u)^{\alpha-1}_\mp du,$

Now we are prepared to characterize the poles and integrability of z_H and its respective $s-$derivatives.

Lemma 3.2. For $t > s$ it holds:

(i) $z_H(t,s) = c_H\, s^{H-\frac{1}{2}}\, G_H(\frac{s}{t})\, 1_{[0,t]}(s) = c_H\, t^{H-\frac{1}{2}}\, s^{-(H-\frac{1}{2})}\, (t-s)^{H-\frac{1}{2}}\, \overline{G}_H(\frac{s}{t})\, 1_{[0,t]}(s)$,

$\forall t \in (0,1]$: $z_H(t,s)$ has a pole of order $H - \frac{1}{2}$ at $s = 0$,

(ii) $\frac{\partial}{\partial s} z_H(t,s) = c_H\, ((H-\frac{1}{2})\, s^{H-\frac{3}{2}}\, G_H(\frac{s}{t}) - t^{H+\frac{1}{2}}\, s^{-(H+\frac{1}{2})}\, (t-s)^{H-\frac{3}{2}})\, 1_{[0,t]}(s)$

$= c_H\, t^{H-\frac{1}{2}}\, s^{-(H+\frac{1}{2})}\, (t-s)^{-(\frac{3}{2}-H)}\, (\, (H-\frac{1}{2})\, (t-s)\, \overline{G}_H(\frac{s}{t}) - t\,)\, 1_{[0,t]}(s),$

$\forall t \in (0,1]$: $\frac{\partial}{\partial s} z_H(t,s)$ has a pole of order $H + \frac{1}{2}$ at $s = 0$ and a pole of order $\frac{3}{2} - H$ at $s = t$,

(iii) $\forall\, \beta \in (1,\infty)$: $\frac{\partial}{\partial s} z_H(t,s) \notin \mathcal{L}^\beta([0,1])$,

(iv) $\forall\, \alpha \in (0,1)$: $D^\alpha_+(s)\, z_H(t,s) = c_H t^{2(H-\frac{1}{2})} s^{-(H-\frac{1}{2})-\alpha} \int\limits_0^1 v^{-(H-\frac{1}{2})}(1-v)^{-\alpha}\cdot$

$((H+\frac{1}{2}-\alpha)(t-s)^{\frac{3}{2}-H}(1-\frac{s}{t}v)^{H-\frac{1}{2}}\bar{G}_H(\frac{s}{t}v) - t^{\frac{3}{2}-H}(1-\frac{s}{t}v)^{H-\frac{3}{2}})dv,$

$D^\alpha_+(s)\, z_H(t,s)$ has a pole of order $(H-\frac{1}{2})+\alpha$ at 0, for $\alpha \in [H-\frac{1}{2}, \frac{3}{2}-H)$ it has a pole at $s = t$. The pole is $(\frac{1}{\alpha-(H-\frac{1}{2})} \wedge \frac{2}{\frac{3}{2}-H+\alpha}$-integrable.

(v) For $\alpha \in [H+\frac{1}{2}, \frac{5}{2}-3H)$ $\forall\, q \in (1, \frac{2}{\frac{3}{2}-H+\alpha}) : D^\alpha_+(s)\, z_H(t,s) \in \mathcal{L}^q([0,1]),$

Otherwise, $\forall\, q \in (1, (H-\frac{1}{2}+\alpha)^{-1}) : D^\alpha_+(s)\, z_H(t,s) \in \mathcal{L}^q([0,1]),$

$\forall\, q \in [(H-\frac{1}{2}+\alpha)^{-1}, \infty) : D^\alpha_+(s)\, z_H(t,s) \notin \mathcal{L}^q([0,1]).$

The previous lemma allows us to derive integration by parts formulas.

Lemma 3.3. Let $t \in [0,1]$, $f \in \mathcal{C}([0,1])$.

(i) $\int\limits_0^t z_H(t,s)\, f(s)\, ds = -\int\limits_0^t \frac{\partial}{\partial s} z_H(t,s) \int\limits_0^s f(u)\, du\, ds,$

(ii) $\int\limits_0^t z_H(t,s)\, f(s)\, ds = \int\limits_0^t D^\alpha_+(s) z_H(t,s)\, I^\alpha_-(s)\, f(s)\, ds, \quad \forall\, \alpha \in (0,1).$

The integration by parts formulas imply the next convergence criterion.

Note, that the weight function w compensates a non-integrable pole of $D_+^\alpha(s) z_H(t, \_)$ at $s = 0$.

Lemma 3.4. Let $w(\_;\beta) \in \mathcal{C}([0,1],(0,1))$ s.t. $\forall \beta \in (0,1) \lim_{s \downarrow 0} w(s) s^{-\beta} \in \mathbb{R}$, $f \in \mathcal{C}([0,1])$, $M_H = \{(\alpha, q) | \alpha \in [H - \frac{1}{2}, \frac{3}{2} - H), q \in (1, \frac{1}{H - \frac{1}{2} + \alpha} \wedge \frac{2}{\frac{3}{2} - H + \alpha}\} \cup (0, H - \frac{1}{2}) \times (1, \infty)$, $M_H^w = (0, \frac{3}{2} - H) \times (1, \infty) \setminus M_H$.

(*i*) $\forall (\alpha, q) \in M_H, p = \frac{q}{q-1}$:

$$|\int_0^t z_H(t,s)\, f(s)\, ds\,| \le \sup_{t \in [0,1]} \{\, \| D_+^\alpha z_H(t, \_) \|_{\mathcal{L}^q([0,1])} \,\} \, \| I_+^\alpha f \|_{\mathcal{L}^p([0,1])},$$

(*ii*) $\forall (\alpha, q) \in M_H^w, \beta \in (0, \alpha) : (\alpha - \beta, q) \in M_H, p = \frac{q}{q-1}$:

$$|\int_0^t z_H(t,s)\, f(s)\, ds\,| \le \sup_{t \in [0,1]} \{\, \left\| \frac{D_+^\alpha z_H(t,s)}{w(s;\beta)} \right\|_{\mathcal{L}^q([0,1])} \,\} \, \| w(\_;\beta)\, I_+^\alpha f \|_{\mathcal{L}^p([0,1])},$$

Example. Consider $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N < \ldots$ such that $\alpha_k \uparrow \infty$. Then φ with $\varphi_k = t^{\alpha_k}$, $k \in \mathbb{N}$, is a complete system. One observes then that for all $k \in \mathbb{N}$: $\int_0^t z_H(t,s) \varphi_k \, ds = -\frac{1}{1+\alpha_k} \int_0^t \frac{\partial}{\partial s} z_H(t,s)\, s^{1+\alpha_k}\, ds$

$$Err_2(\sum_{i=N}^{\infty} \psi_i Y_i) = \tilde{\tilde{\psi}}_{N,\infty}^\top D_{N,\infty} C(\varphi)^{-1} D_{N,\infty} \tilde{\tilde{\psi}}_{N,\infty}$$

where $\tilde{\tilde{\psi}}_{N,\infty} = -\int_0^t s \frac{\partial}{\partial s} z_H(t,s) \varphi_{N,\infty}(s)\, ds$ has finite components and $D_{N,\infty} = diag(\frac{1}{1+\alpha_{N+1}}, \ldots)$ contains convergence information. However, this error representation could not be exploited to find an explicite error rate.

We want to conclude this section with the remark that finding an complete system which allows the explicite computation of truncation points for given error bounds is not easy and will be object of further studies.

Proof lemma 3.1.

(1) Let $f_H(v) := (1-v)^{H - \frac{3}{2}}$, $g_H(v) := v^{-2H}$. As $H - \frac{3}{2} > -1$ $\forall \epsilon \in (0,1)$: $f_H, f_H g_H \in \mathcal{L}([\epsilon, 1])$. As $f_H, g_H \in \mathcal{C}^1((0,1), \mathbb{R}_+)$, $G_H \in \mathcal{C}^1((0,1])$ and $G_H'(v) = -f_H(v) g_H(v)$. $f_H, g_H \in \mathcal{C}^1((0,1),\mathbb{R}_+)$ and the Leibnitz-rule show (i).

(2) Let $\beta \in \mathbb{R} \setminus \{0\}$. (ii), (iii) follow from limit considerations (l'Hospital).

Proof lemma 3.2.

(1) $z_H(t,s) \;=\; c_H\, s^{H-\frac{1}{2}} \int_s^t \left(\frac{u}{s} - 1\right)^{H - \frac{3}{2}} \left(\frac{u}{s}\right)^{H - \frac{1}{2}} \frac{1}{s}\, du\, 1_{[0,t]}(s).$

Apply the variable transformation $v = s/u$ and the definition of G_H.

(2) $\frac{\partial}{\partial s} z_H(t,s) \underset{(ii)}{=} c_H\, (\, (H - \frac{1}{2})\, s^{H - \frac{3}{2}}\, G_H(\frac{s}{t}) + s^{H - \frac{1}{2}}\, G_H'(\frac{s}{t})\,)\, 1_{[0,t]}(s)$

Then trivial term manipulations and the definition of G, $\overline{G}$ show (ii).

(3) $\frac{\partial}{\partial s} z_H(t,s)$ has a pole of order $H + \frac{1}{2} > 1$ at $s = 0$.

(4) $\int_0^s v^{H-\frac{1}{2}} G_H(\frac{v}{t})(s-v)^{-\alpha} dv \underset{v = s\bar{v}}{=} s^{H + \frac{1}{2} - \alpha} \int_0^1 \bar{v}^{H - \frac{1}{2}} (1 - \bar{v})^{-\alpha} G_H(\frac{s}{t}\bar{v}) d\bar{v},$

Use the product rule, $G'_H(\frac{s}{t}\bar{v}) = (1-\frac{s}{t}\bar{v})^{H-\frac{3}{2}}\left(\frac{s}{t}\bar{v}\right)^{-2H}$, collect terms.

(5) $v^{H-\frac{1}{2}}(1-v)^{-\alpha}(1-\frac{s}{t}v)^{H-\frac{3}{2}}$ determines the pole at $s=t$. Its product representation (Beta-kernels), Hölder-inequalities restrict the integrability order q: $\alpha-1<\delta$, $1\wedge\frac{1}{1-\alpha+\delta}<q<\frac{2}{\frac{3}{2}-H+\delta}$, $q<\frac{1}{\delta}$ iff $0<\delta$. Shows (iv).

(6) Checking the poles and integrability condition from (8) shows (v).

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