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Path Space Large Deviations of a Large Buffer with Gaussian Input Traffic

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Abstract. We consider a queue fed by Gaussian traffic and give conditions on the input process under which the path space large deviations of the queue are governed by the rate function of the fractional Brownian motion. As an example we consider input traffic that is composed of of independent streams, each of which is a fractional Brownian motion, having different Hurst indices.

Keywords: fluid queue, Gaussian input, large deviations

1. Introduction

A teletraffic model based on the fractional Brownian motion (fBm) was introduced in [9]. The asymptotics of the busy periods of such model were studied in [10]. We generalise the results given in the latter article to a setting where the input traffic is not quite the fBm but behaves as one in the large time scales.

As input traffic consider a zero mean Gaussian process $Z = (Z_t\colon\ t \in \mathbb{R})$ with stationary increments and regularly varying variance with index $2H$, i.e.

$$\mathbf{Var}\, Z_t = L(t)|t|^{2H}.$$

Here $H \in (0, 1)$ and L is an even function satisfying

$$\lim_{\alpha\to\pm\infty} \frac{L(\alpha t)}{L(\alpha)} = 1$$

for all positive t, i.e. L is slowly varying at infinity. Moreover, assume that $Z_0 = 0$, i.e. $\lim_{t\to 0} |t|^{2H} L(t) = 0$. Note that if $L \equiv 1$ the process Z is an fBm, i.e. the (upto a multiplicative constant) unique Gaussian process with stationary increments and self-similarity index H.

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The *normalised Gaussian storage* is the process

$$V_t := \sup_{-\infty < s \leqslant t} \big(Z_t - Z_s - (t - s)\big).$$

Thus V is a non-negative stationary process indicating the storage occupancy when the storage is fed by the process Z and the service rate is one (this is no restriction if the effective service rate is constant c, since we can always consider an input process $c^{-1}Z$).

We are interested in the distribution of the excursions of V, i.e. the busy periods of the storage, and the distribution of V_0, i.e. the queue length. We study them in the so-called large deviations framework. In particular, we show that the large deviations asymptotics of the busy periods and the queue length are governed by the same rate function as in the case when the queue is fed by an fBm.

The paper is organised as follows. Section 2 is devoted to the technicalities needed to invoke the large deviations machinery. In particular, lemmas 2.7 and 2.8 provide us sufficient conditions (referred to as assumption C and assumption B, respectively) for the slowly varying function L for the results of section 3 to hold. The main result, theorem 3.5, is stated in section 3.

2. Technicalities

2.1. Weak convergence

Define a family of processes $(Z^{(\alpha)}\colon\ \alpha \geqslant 1)$ by setting

$$Z_t^{(\alpha)} := \frac{1}{\alpha^H \sqrt{L(\alpha)}} Z_{\alpha t}. \tag{2.1}$$

Proposition 2.1. The family (2.1) converges to an fBm in finite dimensional distributions.

Proof. Consider first a single time point $t \in \mathbb{R}$. Now the random variable $Z_t^{(\alpha)}$ is Gaussian with mean zero and variance $(L(\alpha t)/L(\alpha))t^{2H}$. Since L is slowly varying at infinity it follows that

$$\lim_{\alpha \to \infty} \mathbf{Var}\, Z_t^{(\alpha)} = t^{2H}, \tag{2.2}$$

i.e. the marginal distributions of $Z^{(\alpha)}$ converge to those of an fBm. Since each $Z^{(\alpha)}$ is a process with stationary increments the limit process must be one with stationary increments also. Moreover, the asymptotic covariances are determined by the asymptotic variances, viz.

$$\begin{aligned}\mathbf{Cov}\big(Z_s^{(\alpha)}, Z_t^{(\alpha)}\big) &= \frac{1}{2}\big(\mathbf{Var}\, Z_s^{(\alpha)} + \mathbf{Var}\, Z_t^{(\alpha)} - \mathbf{Var}\, Z_{t-s}^{(\alpha)}\big) \\ &\to \frac{1}{2}\big(t^{2H} + s^{2H} - |t - s|^{2H}\big)\end{aligned}$$

as α tends to infinity by virtue of (2.2). But this is the covariance function of a fBm. The claim follows, since in the centred Gaussian case the covariance function determines the distribution. □

Having proved the finite dimensional convergence let us turn to the weak convergence. To this end denote by $\mathcal{C}(X)$ the space of continuous real valued functions over the set X equipped with the supremum norm. Consider the weighted space Ω defined by

$$\Omega := \left\{\omega \in \mathcal{C}(\mathbb{R})\colon\ \omega(0) = 0,\ \lim_{|t|\to\infty} \frac{\omega(t)}{1+|t|} = 0\right\}$$

equipped with the norm

$$\|\omega\|_\Omega := \sup_{t\in\mathbb{R}} \frac{|\omega(t)|}{1+|t|}.$$

Remark 2.2. The space Ω is a separable Banach space. Hence, the weak convergence is equivalent to finite dimensional convergence with tightness (cf. Billingsley [2], theorems 5.1 and 5.2).

We have the following characterisations of pre-compact sets and tightness in Ω. For the proofs we refer to lemmas 1 and 3 of [4].

Lemma 2.3. A set B is pre-compact in Ω if and only if

(i) for any $T > 0$ the set

$$\{\omega|_{[-T,T]}\colon\ \omega \in B\}$$

is pre-compact in $\mathcal{C}([-T, T])$,

(ii)

$$\lim_{T\to\infty} \sup_{\omega\in B} \sup_{|t|\geqslant T} \frac{|\omega(t)|}{1+|t|} = 0.$$

The Ascoli–Arzelà equicontinuity characterisation of pre-compact sets in the space $\mathcal{C}([-T, T])$ yields the following tightness characterisation.

Lemma 2.4. The family (2.1) is tight in Ω if and only if

(i) for any $\varepsilon, T > 0$

$$\lim_{\delta\downarrow 0} \sup_{\alpha\geqslant 1} \mathbf{P}\Big(\sup_{\substack{s,t\in[-T,T]\\ |t-s|\leqslant\delta}} \big|Z_t^{(\alpha)} - Z_s^{(\alpha)}\big| > \varepsilon\Big) = 0,$$

(ii) for any $\varepsilon > 0$

$$\lim_{T\to\infty} \sup_{\alpha\geqslant 1} \mathbf{P}\left(\sup_{|t|\geqslant T} \frac{|Z_t^{(\alpha)}|}{1+|t|} > \varepsilon\right) = 0.$$

So far we have assumed nothing about Z that would make its sample paths to belong to the space Ω or even to be continuous. Just assuming that $Z \in \Omega$ would not give us enough to prove the tightness and later the so-called exponential tightness. Moreover, since we are considering functions over a non-compact parameter space there is a need for machinery that is more powerful than the Kolmogorov criterion.

Assumptions C and B below will ensure that Z has sample paths in Ω almost surely. Moreover, they provide us quantitative information on the regularity of the sample paths of Z needed to check the conditions (i) and (ii) of lemma 2.4. Assumption C concerns the continuity of Z ensuring equicontinuity of the family $(Z^{(\alpha)}\colon\ \alpha \geqslant 1)$ on any compact interval $[-T, T] \subset \mathbb{R}$. Assumption B concerns the boundedness of Z_t as t tends to infinity. (Actually, assumption C is included in assumption B(ii).)

Let us introduce the so-called metric entropy machinery. Define a majorising variance

$$\bar{\sigma}^2(t) := \sup_{0<s<t} \sup_{\alpha \geqslant 1} \frac{L(\alpha s)}{L(\alpha)} s^{2H}$$

and the associated "metric entropy integral"

$$J(\kappa, T) := \int_0^{\kappa} \left(\ln\left(\frac{T}{2\bar{\sigma}^{(-1)}(u)} + 1 \right) \right)^{1/2} \mathrm{d}u.$$

The name metric entropy integral comes from the following. Consider the space $[0, T]$ equipped with the seminorm $\bar{\sigma}$. Let $N_{\bar{\sigma}}(u)$ denote the least number of closed balls with radius u needed to cover the space $[0, T]$. The logarithm of $N_{\bar{\sigma}}$, denoted by $H_{\bar{\sigma}}$, is called the *metric entropy* of the space $[0, T]$ under the pseudometric induced by $\bar{\sigma}$. Obviously, we have

$$H_{\bar{\sigma}}(u) \leqslant \ln\left(\frac{T}{2\bar{\sigma}^{(-1)}(u)} + 1 \right).$$

For details of metric entropy we refer to Buldygin and Kozachenko [3].

Assumption C. The integral $J(\bar{\sigma}(T), T)$ is finite for every $T > 0$.

Assumption B. There exists a sequence $(x_k\colon\ k \in \mathbb{N})$ increasing to infinity such that

(i) for all $T \in \mathbb{N}$

$$d_T := \sum_{k=T}^{\infty} c(x_k)\bar{\sigma}(x_k) < \infty,$$

(ii)

$$\sum_{k=1}^{\infty} c(x_k) J\big(\bar{\sigma}(\Delta x_k), \Delta x_k\big) < \infty,$$

where we have denoted $\Delta x_k := x_{k+1} - x_k$ and $c(x) = 1/(1 + x)$.

Remark 2.5. (i) One can replace the majorising variance $\bar{\sigma}$ by any increasing function σ such that $\bar{\sigma}(u) \leqslant \sigma(u)$ for all u.

(ii) Assumption C is satisfied if for some $\varepsilon > 0$, such that $\bar{\sigma}^{(-1)}(\varepsilon) < 1$, the "Dudley integral"

$$\int_0^{\varepsilon} \left|\ln \bar{\sigma}^{(-1)}(u)\right|^{1/2} \mathrm{d}u$$

converges.

Theorem 2.6. If the assumptions C and B hold then the family (2.1) converges weakly in Ω to an fBm.

Theorem 2.6 follows from the lemmas 2.7 and 2.8 below, providing us the conditions (i) and (ii) of lemma 2.4, respectively.

Lemma 2.7. If assumption C holds then the $Z^{(\alpha)}$'s are almost surely continuous on $[0, T]$. Moreover, for

$$\delta > \frac{8}{p(1-p)} J\big(p\bar{\sigma}(\varepsilon), T\big)$$

we have

$$\mathbf{P}\Big(\sup_{\substack{s,t\in[0,T]\\|t-s|<\varepsilon}} \big|Z_t^{(\alpha)} - Z_s^{(\alpha)}\big| > \delta\Big) \leqslant 2\exp\bigg(-\frac{(\delta - 8/(p(1-p))J(\bar{\sigma}(\varepsilon), T))^2}{4\bar{\sigma}^2(\varepsilon)c(p)}\bigg),$$

where $\varepsilon \leqslant T$, and $p \in (0, 1)$.

Lemma 2.8. If assumption B holds then for any integer $T \geqslant 0$ and $\varepsilon > B_T(p)$

$$\mathbf{P}\Big(\sup_{t\geqslant x_T} \big|c(t)Z_t^{(\alpha)}\big| > \varepsilon\Big) \leqslant 2\exp\bigg(-\frac{(\varepsilon - B_T(p))^2}{2A_T(p)}\bigg),$$

where

$$B_T(p) = \frac{1}{p(1-p)} \sum_{k=T}^{\infty} c(x_k) J\big(p\bar{\sigma}(\Delta x_k), \Delta x_k\big),$$

$$A_T(p) = \frac{d_T^2}{(1-p)}\bigg(1 + \frac{2\beta_T^2}{p(1-p)}\bigg),$$

$$\beta_T = \sup_{k\geqslant T} \frac{\bar{\sigma}(\Delta x_k)}{\bar{\sigma}(x_k)},$$

and $p \in (0, 1)$.

Lemma 2.7 can be proved like theorem 3.4.2 of Buldygin and Kozachenko [3]. Lemma 2.8 is a corollary of Kozachenko and Vasilik [8], lemma 4.2.

The idea of the proofs is the following: Choose any $p \in (0,1)$ and set $\varepsilon_n = \bar{\sigma}^{(-1)}(p^n \sup_{t\in[0,T]} \bar{\sigma}(t))$ for the case of lemma 2.7 and $\varepsilon_{nk} = \bar{\sigma}^{(-1)}(p^n \bar{\sigma}(\Delta x_k))$ for the case of lemma 2.8. Let S_n be a minimal ε_n-net of the space $[0,T]$. Then $S = \bigcup_n S_n$ is a separability set of $[0,T]$. Consider mappings $\alpha_n : S \to S_N$, $n = 0, 1, \ldots,$ where $\alpha_n(t) = t$ is $t \in S_n$ and otherwise $\alpha_n(t)$ is a point in S_n satisfying $|t - \alpha_n(t)| < \varepsilon_n$. It is easy to see that $Z_t - Z_{\alpha_n(t)} \to 0$ as $n \to \infty$ with probability one uniformly in t.

Let t be an arbitrary point from S. For any $m \geqslant 1$, denote $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m), \ldots, t_1 = \alpha_1(t_2)$.

For any $t, s \in S$ there exists such integers m and n that $t \in S_m$ and $s \in S_n$. Let us take any $\varepsilon > 0$ and choose such a k that $\bar{\sigma}(\varepsilon_k) < \bar{\sigma}(\varepsilon) < \bar{\sigma}(\varepsilon_{k-1})$ (i.e., $p^k \sup_{t\in[0,T]} \bar{\sigma}(t) < \bar{\sigma}(\varepsilon) < p^{k-1} \sup_{t\in[0,T]} \bar{\sigma}(t)$).

For any $t, s \in S$ we have

$$|Z_t - Z_s| \leqslant 2 \sum_{\ell=k}^{\infty} \max_{t\in S_{\ell+1}} |Z_t - Z_{\alpha_\ell(t)}| + |Z_{t_k} - Z_{s_k}|,$$

$$|Z_{t_k} - Z_{s_k}| \leqslant 2 \sum_{\ell=k}^{\infty} \max_{t\in S_{\ell+1}} |Z_t - Z_{\alpha_\ell(t)}| + |Z_t - Z_s|.$$

So, by choosing certain numbers $q_n > 1$ we obtain by using the Hölder inequality that

$$\begin{aligned}
&\mathbf{E}\exp\bigl(\lambda|Z_{t_k} - Z_{s_k}|\bigr) \\
&\leqslant \bigl(\mathbf{E}\exp\bigl(\lambda q_0|Z_t - Z_s|\bigr)\bigr)^{1/q_0} \prod_{l=k}^{\infty} \Bigl(\mathbf{E}\exp\Bigl(2\lambda q_{\ell-k+1} \max_{t\in S_{\ell+1}} |Z_t - Z_{\alpha_l(t)}|\Bigr)\Bigr)^{1/(q_{\ell-k+1})} \\
&\leqslant \left(\exp\left(\frac{1}{2}\bigl(\lambda q_0 \bar{\sigma}(\varepsilon)\bigr)^2\right)\right)^{1/q_0} \prod_{\ell=k}^{\infty} \bigl(N_{\bar{\sigma}}(\varepsilon_{\ell+1}) \exp\bigl(2\bigl(\lambda q_{\ell-k+1}\bar{\sigma}(\varepsilon_\ell)\bigr)^2\bigr)\bigr)^{1/(q_{\ell-k+1})} \\
&\leqslant \exp\left(\frac{1}{2q_0}\bigl(\lambda q_0 \bar{\sigma}(\varepsilon)\bigr)^2 + \sum_{\ell=k}^{\infty} \frac{1}{q_{l-k+1}} \bigl(H_{\bar{\sigma}}(\varepsilon_{\ell+1}) + 2\bigl(\lambda q_{\ell-k+1}\bar{\sigma}(\varepsilon_\ell)\bigr)^2\bigr)\right)
\end{aligned}$$

for any $\lambda > 0$.

Setting $q_0 = 1/(1-p)$ and

$$q_n = \frac{\sqrt{(2\lambda\bar{\sigma}(\varepsilon_{k-1})/(1-p))^2 + 2H_{\bar{\sigma}}(\varepsilon_{n+k})}}{2\lambda\bar{\sigma}(\varepsilon_{n+k-1})},$$

$n = 1, 2, \ldots,$ we obtain that

$$\begin{aligned}
&\mathbf{E}\exp\Bigl(\lambda \sup_{|t-s|<\varepsilon} |Z_t - Z_s|\Bigr) \\
&\leqslant \exp\left(\lambda^2\bar{\sigma}^2(\varepsilon)\left(\frac{1}{2(1-p)} + \frac{4}{p(1-p)^2}\right) + \frac{8\lambda}{p(1-p)} J\bigl(p\bar{\sigma}(\varepsilon), T\bigr)\right).
\end{aligned}$$

Using the Chebychev's inequality one gets the upper bound of lemma 2.7 using this method. The result of lemma 2.8 comes using the method above by considering compact subsets $[x_{k+1}, x_k]$, $k = 1, 2, \ldots$, separately.

Proof of theorem 2.6. Lemma 2.7 corresponding to assumption C and the symmetry of the process Z gives us the condition (i) of lemma 2.4 since for all $T > 0$

$$\lim_{\varepsilon \downarrow 0} \bar{\sigma}(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} J\big(\bar{\sigma}(\varepsilon), T\big) = 0.$$

Also, lemma 2.8 corresponding to assumption B together with the symmetry of Z gives us the condition (ii) of lemma 2.4 since

$$\lim_{T \to \infty} B_T(p) = 0 \quad \text{and} \quad \lim_{T \to \infty} A_T(p) = 0$$

for all $p \in (0, 1)$. Hence, the family (2.1) is tight. The claim follows now from proposition 2.1. □

Since it is in practise somewhat tedious to check the assumptions C and B we provide the following proposition.

Proposition 2.9. Suppose that the majorising variance $\bar{\sigma}$ satisfies

$$\bar{\sigma}(t) \leqslant \frac{D}{(\ln(1 + 1/t))^{\varepsilon}}$$

for some $\varepsilon \in (1/2, 1)$. Then the assumptions C and B are satisfied.

Proof. We may suppose that $T > 2$. Now

$$\frac{T}{2\bar{\sigma}^{(-1)}(u)} + 1 \leqslant \frac{T(\exp((D/u)^{1/\varepsilon}) - 1)}{2} + 1 \leqslant \frac{T}{2} \exp\left(\left(\frac{D}{u}\right)^{1/\varepsilon}\right)$$

and

$$\left(\ln\left(\frac{T}{2\bar{\sigma}^{(-1)}(u)} + 1\right)\right)^{1/2} \leqslant \left(\ln \frac{T}{2} + \left(\frac{D}{u}\right)^{1/\varepsilon}\right)^{1/2} \leqslant \left(\ln \frac{T}{2}\right)^{1/2} + \left(\frac{D}{u}\right)^{1/(2\varepsilon)}.$$

Therefore,

$$\begin{aligned} J(\kappa, T) &\leqslant \kappa \left(\ln \frac{T}{2}\right)^{1/2} + D^{1/(2\varepsilon)} \int_0^{\kappa} \frac{1}{u^{1/(2\varepsilon)}} \, \mathrm{d}u \\ &= \kappa \left(\ln \frac{T}{2}\right)^{1/2} + D^{1/(2\varepsilon)} \kappa^{1-1/(2\varepsilon)} \frac{1}{1 - 1/(2\varepsilon)}. \end{aligned} \tag{2.3}$$

In particular, assumption C is satisfied.

Consider then assumption B. By the inequality (2.3) it is enough to show that

$$\sum_{k=1}^{\infty} \frac{1}{1+x_k} \frac{D}{(\ln(1+1/x_k))^{\varepsilon}} < \infty, \tag{2.4}$$

$$\sum_{k=1}^{\infty} \frac{1}{1+x_k} \left(\ln \frac{\Delta x_k}{2}\right)^{1/2} \frac{D}{(\ln(1+1/(\Delta x_k)))^{\varepsilon}} < \infty. \tag{2.5}$$

The convergence of the series (2.5) is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{x_k} (\ln \Delta x_k)^{1/2} (\Delta x_k)^{\varepsilon}. \tag{2.6}$$

Choose $x_k = \mathrm{e}^k$. Thus, $\Delta x_k = \mathrm{e}^k(\mathrm{e}-1)$ and (2.6) converges, since

$$\sum_{k=1}^{\infty} \frac{1}{\mathrm{e}^k} \left(\ln \mathrm{e}^k\right)^{1/2} \left(\mathrm{e}^k\right)^{\varepsilon} = \sum_{k=1}^{\infty} \frac{1}{\mathrm{e}^k} k^{1/2} \mathrm{e}^{k\varepsilon} = \sum_{k=1}^{\infty} k^{1/2} \mathrm{e}^{k(\varepsilon-1)} < \infty.$$

With this choice of (x_k) the series (2.4) also converges. So, assumption B is satisfied. □

Example 2.10. Suppose the input traffic is composed of independent streams, each of which is an fBm, with different Hurst indices, i.e.

$$Z = \sum_{k=1}^{\infty} a_k B^{H_k},$$

where $1 > H_1 > H_2 > H_3 > \cdots$ and $H_k \to 0$ as $k \to \infty$. Assume that

$$\sum_{k=1}^{\infty} a_k^2 H_k^{-2\varepsilon} < \infty$$

for some $\varepsilon > \max(1/2, H_1)$. Now

$$\mathbf{E} Z_t^2 = \sum_{k=1}^{\infty} a_k^2 |t|^{2H_k} = |t|^{2H_1} \sum_{k=1}^{\infty} a_k^2 |t|^{2(H_k - H_1)}.$$

So Z has regularly varying variance with index $2H_1$ with

$$L(t) = \sum_{k=1}^{\infty} a_k^2 |t|^{2(H_k - H_1)}$$

and

$$\bar{\sigma}^2(t) = \sup_{0<s<t} \sup_{\alpha \geqslant 1} \frac{\sum_{k=1}^{\infty} a_k^2 (\alpha s)^{2H_k}}{\sum_{k=1}^{\infty} a_k^2 \alpha^{2H_k}} \leqslant \sup_{\alpha \geqslant 1} \frac{\sum_{k=1}^{\infty} a_k^2 (\alpha t)^{2H_k}}{\sum_{k=1}^{\infty} a_k^2 \alpha^{2H_k}}. \tag{2.7}$$

Using the fact that $1/x < 1/\ln(1+x)$ for all positive x we obtain

$$t^H < \frac{1}{\ln(1+1/t^H)} < \frac{1}{\ln(1+1/t)^H} = \frac{1}{H\ln(1+1/t)}.$$

So, for any $\varepsilon > \max(1/2, H)$ we have

$$t^{2H_k} = t^{(H_k/\varepsilon)2\varepsilon} \leqslant \frac{1}{(H_k/\varepsilon)^{2\varepsilon}(\ln(1+1/t))^{2\varepsilon}}.$$

Using this to (2.7) we obtain

$$\begin{aligned}\bar{\sigma}^2(t) &\leqslant \sup_{\alpha\geqslant 1} \varepsilon^{2\varepsilon} \frac{\sum_{k=1}^{\infty} a_k^2 H_k^{-2\varepsilon} \alpha^{2H_k}}{\sum_{k=1}^{\infty} a_k^2 \alpha^{2H_k}} \frac{1}{(\ln(1+1/t))^{2\varepsilon}} \\ &\leqslant \sup_{\alpha\geqslant 1} \varepsilon^{2\varepsilon} \frac{\sum_{k=1}^{\infty} a_k^2 H_k^{-2\varepsilon} \alpha^{2H_1}}{\sum_{k=1}^{\infty} a_k^2 \alpha^{2H_k}} \frac{1}{(\ln(1+1/t))^{2\varepsilon}} \\ &\leqslant \varepsilon^{2\varepsilon} \frac{1}{a_1^2} \sum_{k=1}^{\infty} a_k^2 H_k^{-2\varepsilon} \frac{1}{(\ln(1+1/t))^{2\varepsilon}}.\end{aligned}$$

It follows now from proposition 2.9 that the process Z satisfies the assumptions C and B.

Remark 2.11. In example 2.10 above we have $\bar{\sigma}^2(t) \geqslant |t|^\beta$ as $t \to 0$, where β is any positive number. Therefore, the Kolmogorov criterion cannot be used to prove tightness in this case.

2.2. Large deviations

Let us briefly define the large deviations framework. For details we refer to Dembo and Zeitouni [5].

Definition 2.12. A rate function I is a lower semicontinuous mapping $I : \Omega \to [0,\infty]$. If the level sets $\{\omega\colon I(\omega) \leqslant a\}$, $a \in \mathbb{R}$, are compact then the rate function I is called good. A set $A \subset \Omega$ is called good (with respect to I) if $\inf_{\omega\in A^\circ} I(\omega) = \inf_{\omega\in\overline{A}} I(\omega)$. A scale is a mapping $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\alpha\to\infty} v(\alpha) = \infty$.

Remark 2.13. In what follows the rate functions are always good.

Definition 2.14. A scaled family $((X^{(\alpha)}, v(\alpha))\colon \alpha \geqslant 1)$ of Ω-valued random variables satisfies the large deviations principle (LDP) on Ω if there exists a rate function $I : \Omega \to [0,\infty]$ such that for all closed $F \subset \Omega$ and open $G \subset \Omega$

$$\begin{aligned}\limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbf{P}\big(X^{(\alpha)} \in F\big) &\leqslant -\inf_{\omega\in F} I(\omega), \\ \liminf_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbf{P}\big(X^{(\alpha)} \in G\big) &\geqslant -\inf_{\omega\in G} I(\omega).\end{aligned}$$

Remark 2.15. In the case of good sets the upper and lower bounds above coincide giving a proper limit.

Motivated by [10] we define the scale v as

$$v(\alpha) := \frac{\alpha^{2-2H}}{L(\alpha)}.$$

Our aim is to show that the scaled family

$$\left(\left(\frac{1}{\sqrt{v(\alpha)}} Z^{(\alpha)}, v(\alpha)\right):\ \alpha \geqslant 1\right) \tag{2.8}$$

satisfies the LDP on Ω.

In order to prove the LDP on the function space Ω we follow the usual approach by first proving the LDP in the finite dimensional spaces $\mathbb{R}^d$ for all $d \in \mathbb{N}$. Then we lift the LDP to Ω by means of the so called inverse contraction principle. For details of this approach see Dembo and Zeitouni [5], chapters 2 and 4.

Fix a dimension $d \in \mathbb{N}$ and a vector $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$. Denote by $\mathbf{Z}^{(\alpha)}(\mathbf{t})$ the projection

$$\mathbf{Z}^{(\alpha)}(\mathbf{t}) := \left(Z^{(\alpha)}_{t_1}, \ldots, Z^{(\alpha)}_{t_d}\right).$$

The vector $\mathbf{Z}^{(\alpha)}(\mathbf{t})$ is a multivariate normal random variable. Let $\Gamma_\alpha(\mathbf{t})$ denote its covariance matrix. Similarly let $\Gamma(\mathbf{t})$ denote the covariance matrix of the corresponding projection of the fBm and define the rate function $I(\cdot, \mathbf{t})$ by

$$I(\mathbf{x}, \mathbf{t}) := \frac{1}{2}\|\mathbf{x}\|^2_{\Gamma^{-1}(\mathbf{t})} := \frac{1}{2}\left\langle \Gamma^{-1}(\mathbf{t})\mathbf{x}, \mathbf{x}\right\rangle.$$

Remark 2.16. Recall the reproducting kernel Hilbert space (RKHS) associated to a covariance function, say R, of a centred Gaussian family $(X_t\colon\ t \in T)$ (for details see, e.g., Alder [1], p. 66). Let $\mathcal{S}(R)$ be the set of functions of the form

$$f = \sum_{k=1}^{n} f_k R(t_k, \cdot),$$

$f_k \in \mathbb{R}$, and define an inner product on S by

$$\langle f, g\rangle_{\mathcal{H}} := \sum_{k=1}^{n}\sum_{\ell=1}^{m} f_k g_\ell R(t_k, t_\ell).$$

The closure of $\mathcal{S}(R)$ under the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, denoted by $\mathcal{H}(R)$, is called the RKHS associated to the covariance R. The inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ has the “reproducing kernel” property

$$f(t) = \left\langle f, R(t, \cdot)\right\rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}(R)$.

Now, it is easy to see that the norm $\|\cdot\|_{\Gamma^{-1}(\mathbf{t})}$ is the RKHS norm associated to the covariance $\Gamma(\mathbf{t})$ (there is no need to take the closure in this case, of course).

Lemma 2.17. For all $d \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{R}^d$ the scaled family

$$\left(\left(\frac{1}{\sqrt{v(\alpha)}}\mathbf{Z}^{(\alpha)}(\mathbf{t}),\, v(\alpha)\right):\ \alpha \geqslant 1\right)$$

satisfies LDP in $\mathbb{R}^d$ with the good rate function $I(\cdot, \mathbf{t})$.

Proof. Let $d \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{R}^d$ be fixed and omitted in the notation. For any $\mathbf{u} \in \mathbb{R}^d$ set

$$\Lambda_\alpha(\mathbf{u}) := \ln \mathbf{E} \exp\left\langle \mathbf{u},\, \frac{\sqrt{L(\alpha)}}{\alpha^{1-H}}\mathbf{Z}^{(\alpha)}\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$. Now

$$\Lambda_\alpha\left(\frac{\alpha^{2-2H}}{L(\alpha)}\mathbf{u}\right) = \frac{1}{2}\frac{\alpha^{2-2H}}{L(\alpha)}\langle\Gamma_\alpha\mathbf{u},\mathbf{u}\rangle.$$

Since $Z^{(\alpha)}$ converges weakly to the fBm, we have

$$\langle\Gamma_\alpha\mathbf{u},\mathbf{u}\rangle \to \langle\Gamma\mathbf{u},\mathbf{u}\rangle,$$

for all $\mathbf{u} \in \mathbb{R}^d$. Hence

$$\Lambda(\mathbf{u}) := \lim_{\alpha\to\infty}\frac{1}{v(\alpha)}\Lambda_\alpha\big(v(\alpha)\mathbf{u}\big)$$

exists and is finite for all $\mathbf{u} \in \mathbb{R}^d$. In particular,

$$\Lambda(\mathbf{u}) = \frac{1}{2}\langle\Gamma\mathbf{u},\mathbf{u}\rangle.$$

Moreover, the Fenchel–Legendre transform Λ^* of Λ is of the familiar Gaussian form

$$\Lambda^*(\mathbf{x}) = I(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2_{\Gamma^{-1}} := \frac{1}{2}\big\langle\Gamma^{-1}\mathbf{x},\mathbf{x}\big\rangle,$$

where $\|\cdot\|_{\Gamma^{-1}}$ is the norm on $\mathbb{R}^d$ induced by the covariance matrix Γ.

The calculations above show that the assumptions of the Gärtner–Ellis theorem (cf. Dembo and Zeitouni [5], theorem 2.3.6) are satisfied. The claim follows. □

Let us turn now to the infinite dimensional case. Since we have proved the LDP for all finite dimensional projections of Ω, i.e. the $\mathbb{R}^d$'s, we can lift it to Ω endowed with the so-called projective limit topology, or the topology of pointwise convergence. In particular, by the Dawson–Gärtner theorem (cf. Dembo and Zeitouni [5], theorem 4.6.1) we have the next lemma.

Lemma 2.18. Let p be a finite dimensional projection on Ω, i.e. for a function $x \in \Omega$ $p(x)$ is a vector

$$p(x) = \big(x(t_1), x(t_2), \ldots, x(t_d)\big) \in \mathbb{R}^d$$

for some $t_1, t_2, \ldots, t_d \in \mathbb{R}$ and $d \in \mathbb{N}$. Let Π denote the set of all finite dimensional projections on Ω. The family (2.8) satisfies the LDP on Ω equipped with the projective limit topology with the good rate function

$$I(x) = \sup_{p\in\Pi} \frac{1}{2} \big\| p(x) \big\|^2_{\Gamma_p^{-1}}. \tag{2.9}$$

Here Γ_p is the covariance matrix of $p(B^H)$ and B^H is the fBm.

The rate function (2.9) above is that of the fBm in the generalised Schilder's theorem concerning the norm topology in Ω. For details we refer to Deuschel and Stroock [6] and Norros [10]. In particular, note that the rate function I is the same that appears in [10], equation (2.2), where the RKHS norm is used.

It should be noted that this projective limit topology is not a very strong one. Hence, the result of lemma 2.18 is rather inadequate. However, if we show that the family (2.8) is exponentially tight with respect to the $\|\cdot\|_\Omega$ topology the LDP follows from the inverse contraction principle. For details see Dembo and Zeitouni [5], theorem 4.2.4 and corollary 4.2.6.

Definition 2.19. A scaled family $((X^{(\alpha)}, v(\alpha))\colon \alpha \geqslant 1)$ is exponentially tight in Ω if for each $\ell > 0$ there exist pre-compact sets $K_\ell \subset \Omega$ such that

$$\lim_{\ell\to\infty} \limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \ln \mathbf{P}\big(X^{(\alpha)} \notin K_\ell\big) = -\infty$$

holds.

Lemma 2.3 provides us the following characterisation of the exponential tightness in Ω.

Lemma 2.20. The family (2.8) is exponentially tight if and only if

(i) for each $T > 0$

$$\sup_{\alpha\geqslant 1} \frac{1}{v(\alpha)} \ln \mathbf{P}\Big(\sup_{\substack{s,t\in[-T,T]\\ |t-s|\leqslant\delta}} |Z_{\alpha t} - Z_{\alpha s}| \geqslant \alpha\varepsilon \Big) \leqslant -\ell.$$

for all $\varepsilon > 0$ given $\delta = \delta(\varepsilon, \ell)$ is small enough,

(ii)

$$\sup_{\alpha\geqslant 1} \frac{1}{v(\alpha)} \ln \mathbf{P}\left(\sup_{t\geqslant m} \frac{|Z_{\alpha t}|}{1+t} \geqslant \alpha\varepsilon \right) \leqslant -\ell$$

for all $\varepsilon > 0$ given $m = m(\varepsilon, \ell)$ is big enough.

Proof. If the family is exponentially tight, in the sense of definition 2.19, then (i) and (ii) are immediate; to prove the converse note that (i) is just a formulation of the exponential tightness of the family (2.8) on $\mathcal{C}([-T,T])$ using the Ascoli–Arzelà theorem. So we can choose pre-compact sets $K_{\ell,T}$ from $\mathcal{C}([-T,T])$ satisfying

$$\frac{1}{v(\alpha)}\ln \mathbf{P}\biggl(\frac{1}{\sqrt{v(\alpha)}}Z^{(\alpha)} \notin K_{\ell,T}\biggr) \leqslant -\ell 2T \tag{2.10}$$

for all $\alpha \geqslant 1$. Choose $K_{\ell,\infty}$ by (ii) setting ℓ to 2ℓ. For each $\ell > 0$ consider the set

$$K_\ell = \bigcap_{T=1}^{\infty} K_{\ell,T} \ \cap \ K_{\ell,\infty}.$$

By lemma 2.3 it is pre-compact in Ω. Let P_α denote the law of $Z^{(\alpha)}/\sqrt{v(\alpha)}$ under $\mathbf{P}$. Using Boole's inequality and the inequalities (ii) and (2.10) together with the fact that $v(\alpha)$ tends to infinity as α increases we obtain

$$\begin{aligned}
P_\alpha\bigl(K_\ell^c\bigr) &\leqslant \sum_{T=1}^{\infty} P_\alpha\bigl(K_{\ell,T}^c\bigr) + P_\alpha\bigl(K_{\ell,\infty}^c\bigr)\\
&\leqslant \sum_{T=1}^{\infty}\bigl(\mathrm{e}^{-2\ell v(\alpha)}\bigr)^T + \mathrm{e}^{-2\ell v(\alpha)}\\
&= \frac{\mathrm{e}^{-2\ell v(\alpha)}}{1-\mathrm{e}^{-2\ell v(\alpha)}} + \mathrm{e}^{-2\ell v(\alpha)}\\
&\leqslant \mathrm{const}\,\mathrm{e}^{-2\ell v(\alpha)}\\
&\leqslant \mathrm{e}^{-\ell v(\alpha)},
\end{aligned}$$

given sufficiently large ℓ. It follows from this that

$$\lim_{\ell\to\infty}\limsup_{\alpha\to\infty}\frac{1}{v(\alpha)}\ln P_\alpha(K_\ell^c) = -\infty,$$

which is to say that (2.8) is exponentially tight. □

Lemmas 2.7 and 2.8 which provided us tightness also imply the inequalities (i) and (ii) of lemma 2.20, i.e. the exponential tightness. The main result of this section follows.

Theorem 2.21. If the assumptions C and B hold then the scaled family (2.8) satisfies the LDP on Ω with the good rate function I.

Proof. By lemma 2.7 we have

$$\ln \mathbf{P}\Big(\sup_{\substack{s,t\in[0,T]\\|t-s|\leqslant\delta}} |Z_{\alpha t} - Z_{\alpha s}| \geqslant \alpha\varepsilon\Big) = \ln \mathbf{P}\Big(\sup_{\substack{s,t\in[0,T]\\|t-s|\leqslant\delta}} |Z_t^{(\alpha)} - Z_s^{(\alpha)}| \geqslant \sqrt{v(\alpha)}\varepsilon\Big)$$
$$\leqslant \ln 2 - \frac{(\sqrt{v(\alpha)}\varepsilon - \frac{8}{p(1-p)} J(\bar\sigma(\delta), T))^2}{4\bar\sigma(\delta)c(p)}.$$

Hence,

$$\frac{1}{v(\alpha)} \ln \mathbf{P}\Big(\sup_{\substack{s,t\in[0,T]\\|t-s|\leqslant\delta}} |Z_{\alpha t} - Z_{\alpha s}| \geqslant \alpha\varepsilon\Big) \leqslant \frac{\ln 2}{v(\alpha)} - \frac{(\varepsilon - \frac{8}{p(1-p)} J(\bar\sigma(\delta), T)/\sqrt{v(\alpha)})^2}{4\bar\sigma(\delta)c(p)}.$$

Condition (i) of lemma 2.20 follows since $v(\alpha)$ tends to infinity as α increases to infinity and $\bar\sigma(\delta)$ and $J(\bar\sigma(\delta), T)$ can be made arbitrarily small by choosing a small enough δ.

Similarly, by lemma 2.8 we have

$$\frac{1}{v(\alpha)} \ln \mathbf{P}\Big(\sup_{t\geqslant T} \frac{|Z_{\alpha t}|}{1+t} \geqslant \alpha\varepsilon\Big) \leqslant \frac{\ln 2}{v(\alpha)} - \frac{(\varepsilon - B_T(p)/\sqrt{v(\alpha)})^2}{2A_T(p)}.$$

The inequality (ii) of lemma 2.20 follows since $v(\alpha)$ tends to infinity as α increases to infinity and $B_T(p)$ and $A_T(p)$ converge to zero for all $p \in (0, 1)$ as T increases to infinity.

The claim follows now from the symmetry of the process Z. □

3. Large buffer and busy period asymptotics

Consider the storage process

$$V_t(\omega) := \sup_{-\infty<s\leqslant t} \big(Z_t(\omega) - Z_s(\omega) - (t-s)\big)$$
$$= \sup_{-\infty<s\leqslant t} \big(\omega(t) - \omega(s) - (t-s)\big).$$

Note that if the conditions C and B are satisfied then $Z \in \Omega$. Consequently, V is finite.

In the large buffer case the LDP gives us asymptotics of the sets

$$Q_x = \{V_0 \geqslant x\}. \tag{3.1}$$

Lemma 3.1. For any $x \geqslant 1$

$$\mathbf{P}(Z \in Q_x) = \mathbf{P}\Big(\frac{\sqrt{L(x)}}{x^{1-H}} Z^{(x)} \in Q_1\Big),$$

where Q_x is defined by (3.1).

Proof. A simple calculation yields

$$\begin{aligned}\mathbf{P}(Z \in Q_x) &= \mathbf{P}\Big(\sup_{t\leqslant 0}(Z_t - t) \geqslant x\Big)\\ &= \mathbf{P}\Big(\sup_{t\leqslant 0}(Z_{xt} - xt) \geqslant x\Big)\\ &= \mathbf{P}\bigg(\sup_{t\leqslant 0}\bigg(\frac{L(x)}{x^{1-H}}Z_t^{(x)} - t\bigg) \geqslant 1\bigg)\\ &= \mathbf{P}\bigg(\frac{L(x)}{x^{1-H}}Z^{(x)} \in Q_1\bigg).\end{aligned}$$

The claim follows. □

Proposition 3.2. Let Q_x be as in (3.1) and suppose that the assumption C and B are satisfied. Then

$$\lim_{x\to\infty}\frac{L(x)}{x^{2-2H}}\ln \mathbf{P}(Z \in Q_x) = -\inf_{\omega\in Q_1} I(\omega)$$

where I is the good rate function defined in (2.9). Moreover, for the constant $\inf_{\omega\in Q_1} I(\omega)$ we have

$$\inf_{\omega\in Q_1} I(\omega) = \frac{(1-H)^{2H-2}}{2H^{2H}}.$$

Proof. Since Q_1 is a good set (cf. Norros [10]), i.e.

$$\inf_{\omega\in\overline{Q}_1} I(\omega) = \inf_{\omega\in Q_1^\circ} I(\omega),$$

the proposition follows from lemma 3.1 and the LDP. The constant $\inf_{\omega\in Q_1} I(\omega)$ is identified, e.g., in Duffield and O'Connel [7] and in Norros [10]. □

Remark 3.3. The result of proposition 3.2 is not a new one. Indeed, it appears in Duffield and O'Connel [7], equation (54). The conditions of [7] reduce in our case to

$$\lim_{t\to\infty}\inf_{c>(1-H)/H}\frac{(c+1)^2}{c^{2-2H}}\frac{L(ct)}{L(t)} = \frac{(1-H)^{2H-2}}{H^{2H}},$$

i.e. their hypothesis 2.2(ii) on p. 367.

The *busy periods* of V are its positive excursions, i.e. stochastic intervals $[D, E]$ such that $V_t > 0$ for all $t \in (D, E)$ and $V_D = 0 = V_E$. The busy period containing 0 is defined as a stochastic interval

$$[A, B] := \big[\sup\{t \leqslant 0\colon\ V_t = 0\}, \inf\{t \geqslant 0\colon\ V_t = 0\}\big],$$

if $A < 0 < B$. Otherwise the system is not busy at time 0. For a $T > 0$ denote

$$K_T := \{A < 0 < B, B - A > T\} \tag{3.2}$$

the set of paths for which the ongoing busy period at 0 is strictly longer than T.

Lemma 3.4. For any $T \geqslant 1$

$$\mathbf{P}(Z \in K_T) = \mathbf{P}\left(\frac{\sqrt{L(T)}}{T^{1-H}} Z^{(T)} \in K_1\right),$$

where K_T is defined by (3.2).

Proof. Since

$$\begin{aligned}\mathbf{P}(Z \in K_T) &= \mathbf{P}\big(\exists a < 0, b > (a+T)^+ \ \forall t \in (a,b):\ Z_t - Z_a > t - a\big)\\ &= \mathbf{P}\big(\exists a < 0, b > (a+1)^+ \ \forall t \in (a,b):\ Z_{Tt} - Z_{Ta} > Tt - Ta\big)\\ &= \mathbf{P}\big(\exists a < 0, b > (a+1)^+ \ \forall t \in (a,b):\ T^{-1}(Z_{Tt} - Z_{Ta}) > t - a\big)\\ &= \mathbf{P}\left(\frac{\sqrt{L(T)}}{T^{1-H}} Z^{(T)} \in K_1\right),\end{aligned}$$

the lemma follows. □

Finally, for the asymptotics of the busy periods we have the following generalisation of theorem 4.5 of Norros [10].

Theorem 3.5. Let K_T be as is (3.2) and suppose that the assumptions C and B are satisfied. Then

$$\lim_{T\to\infty} \frac{L(T)}{T^{2-2H}} \ln \mathbf{P}(Z \in K_T) = -\inf_{\omega \in K_1} I(\omega),$$

where I is the rate function defined in (2.9). The constant $\inf_{\omega \in K_1} I(\omega)$ lies in the interval $[1/2, c_H^2/2]$, where

$$c_H = \left\{H(2H-1)(2-2H)\mathrm{B}\left(H - \frac{1}{2}, 2 - 2H\right)\right\}^{-1/2}$$

and B is the Beta function

$$\mathrm{B}(\mu, \nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,\mathrm{d}x.$$

Proof. Since K_1 is a good set by proposition 4.3 of [10], the theorem follows from lemma 3.4 and the LDP. The bounds for the constant $\inf_{\omega \in K_1} I(\omega)$ are proved in [10], theorem 4.5. □

Remark 3.6. The exact value of the constant $\inf_{\omega \in K_1} I(\omega)$ is not known. However, Norros [10] indicates how one can numerically find arbitrarily good approximations to it. In particular, if $H > \frac{1}{2}$ then the constant c_H is close to one. Consequenty, $\inf_{\omega \in K_1} I(\omega) \approx \frac{1}{2}$ in this case.

Example 3.7. For the superposition process in example 2.10 we have the scaling function

$$\frac{L(T)}{T^{2-2H}} = \sum_{k=1}^{\infty} a_k^2 T^{2H_k - 2}.$$

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