PATHWISE INTEGRALS AND ITÔ–TANAKA FORMULA FOR GAUSSIAN PROCESSES

TOMMI SOTTINEN AND LAURI VIITASAARI

Abstract. We prove the Itô–Tanaka formula and the existence of pathwise stochastic integrals for a wide class of Gaussian processes. Motivated by financial applications, we define the stochastic integrals as forward-type pathwise integrals introduced by Föllmer and as pathwise generalized Lebesgue–Stieltjes integrals introduced by Zähle. As an application, we illustrate the importance of Itô–Tanaka formula for pricing and hedging of financial derivatives.

1. Introduction

Let Y be a continuous Gaussian stochastic process on the time interval $[0,T]$ and let f be a linear combination of convex functions. We are interested in which generality and for what kind of integrals and notion of local time L_T^a we can obtain the Itô–Tanaka formula

$$f(Y_T) = f(Y_0) + \int_0^T f'_-(Y_u)\,\mathrm{d}Y_u + \frac{1}{2}\int_{-\infty}^{\infty} L_T^a(Y)\, f''(\mathrm{d}a). \tag{1.1}$$

Since we do not assume that Y is a semimartingale, standard stochastic integration theory cannot be applied here and we have to determine in which sense the stochastic integral in (1.1) exists.

Motivated by financial applications we consider pathwise integrals that are generalizations of the financially natural Riemann–Stieltjes integral. These generalizations go back to Young [16]. In particular, we consider the pathwise forward-type Riemann–Stieltjes integral introduced by Föllmer [9] and the pathwise generalized Lebesgue–Stieltjes integrals introduce by Zähle [17] and further studied e.g. by Nualart and Rascanu [11].

Let us note that for Gaussian processes one can also consider Skorokhod integrals and e.g. in the particular case of fractional Brownian motion the Itô–Tanaka formula (1.1) is established in [8]. However, the Skorokhod integrals do not admit an economical interpretation in any obvious way; see [15] for details.

Our work is related to [1, 2, 3, 11], where only the case of fractional Brownian motion were studied. We extend the results to a more general class of Gaussian processes. Bertoin [7] also established the Itô–Tanaka formula (1.1) in the Föllmer sense for a very general class of Dirichlet process. In that

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sense our work is merely a special case of [7]. However, in [7] it was a priori assumed that the local time $L_T^a(Y)$ exists as the Lebesgue density of the occupation measure in the L^2 sense. This actually also implies the existence of the associated Föllmer integral. We do not assume the existence of the density or the integral a priori. Also, the generalized Lebesgue–Stieltjes integrals were not studied in [7].

The rest of the paper is organized as follows. In section 2 we introduce the generalized Lebesgue-Stieltjes integrals and the Föllmer integrals. The main section 3 begins with introducing our assumptions with discussion and examples. Then we prove several fundamental lemmas after which we state and prove our main results on the existence of the generalized Lebesgue–Stieltjes integrals, the Föllmer integrals and mixed Föllmer–generalized Lebesgue–Stieltjes integrals. Then we prove the Itô–Tanaka formula. In Section 4 we discuss the importance of the Itô–Tanaka formula for the hedging and pricing of financial derivatives. Finally, a technical lemma on the level-crossing probabilities of Gaussian processes is given in the appendix.

2. Pathwise Integrals

We recall two notions of pathwise stochastic integrals: the generalized Lebesgue–Stieltjes integral and the Föllmer integral.

Generalized Lebesgue–Stieltjes Integral. The generalized Lebesgue–Stieltjes integral is based on fractional integration and fractional Besov spaces. For details on fractional integration we refer to [13] and for fractional Besov spaces we refer to [11].

We first recall the definitions for fractional Besov norms and Lebesgue–Liouville fractional integrals and derivatives.

Definition 2.1. Fix $0 < \beta < 1$.

(i) The *fractional Besov space* $W_1^\beta = W_1^\beta([0,T])$ is the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{1,\beta} = \sup_{0\le s<t\le T}\left(\frac{|f(t)-f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u)-f(s)|}{(u-s)^{1+\beta}}\,\mathrm{d}u\right) < \infty.$$

(ii) The *fractional Besov space* $W_2^\beta = W_2^\beta([0,T])$ is the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{2,\beta} = \int_0^T \frac{|f(s)|}{s^\beta}\,\mathrm{d}s + \int_0^T\int_0^s \frac{|f(u)-f(s)|}{(u-s)^{1+\beta}}\,\mathrm{d}u\mathrm{d}s < \infty.$$

Remark 2.1. Let $C^\alpha = C^\alpha([0,T])$ denote the space of Hölder continuous functions of order α on $[0,T]$ and let $0 < \epsilon < \beta \wedge (1-\beta)$. Then

$$C^{\beta+\epsilon} \subset W_1^\beta \subset C^{\beta-\epsilon} \quad \text{and} \quad C^{\beta+\epsilon} \subset W_2^\beta.$$

Definition 2.2. Let $t \in [0,T]$. The *Riemann–Liouville fractional integrals* I^β_{0+} and I^β_{t-} of order $\beta > 0$ on $[0,T]$ are

$$(I^\beta_{0+} f)(s) = \frac{1}{\Gamma(\beta)} \int_0^s f(u)(s-u)^{\beta-1}\, \mathrm{d}u,$$
$$(I^\beta_{t-} f)(s) = \frac{(-1)^{-\beta}}{\Gamma(\beta)} \int_s^t f(u)(u-s)^{\beta-1}\, \mathrm{d}u,$$

where Γ is the Gamma-function. The *Riemann–Liouville fractional derivatives* D^β_{0+} and D^β_{t-} are the left-inverses of the corresponding integrals I^β_{0+} and I^β_{t-}. They can be also define via the *Weyl representation* as

$$(D^\beta_{0+} f)(s) = \frac{1}{\Gamma(1-\beta)} \left(\frac{f(s)}{s^\beta} + \beta \int_0^s \frac{f(s)-f(u)}{(s-u)^{\beta+1}}\, \mathrm{d}u \right),$$
$$(D^\beta_{t-} f)(s) = \frac{(-1)^{-\beta}}{\Gamma(1-\beta)} \left(\frac{f(s)}{(t-s)^\beta} + \beta \int_s^t \frac{f(s)-f(u)}{(u-s)^{\beta+1}}\, \mathrm{d}u \right)$$

if $f \in I^\beta_{0+}(L^1)$ or $f \in I^\beta_{t-}(L^1)$, respectively.

Denote $g_{t-}(s) = g(s) - g(t-)$.

The generalized Lebesgue–Stieltjes integral is defined in terms of fractional derivative operators according to the next proposition.

Proposition 2.1. [11] *Let* $0 < \beta < 1$ *and let* $f \in W^\beta_2$ *and* $g \in W^{1-\beta}_1$. *Then for any* $t \in (0,T]$ *the* generalized Lebesgue–Stieltjes integral *exists as the following Lebesgue integral*

$$\int_0^t f(s)\, \mathrm{d}g(s) = \int_0^t (D^\beta_{0+} f)(s)(D^{1-\beta}_{t-} g_{t-})(s)\, \mathrm{d}s$$

and is independent of β.

Remark 2.2. It is shown in [17] that if $f \in C^\gamma$ and $g \in C^{f''}$ with $\gamma + f'' > 1$, then the generalized Lebesgue–Stieltjes integral $\int_0^t f(s)\, \mathrm{d}g(s)$ exists and coincides with the classical Riemann–Stieltjes integral, i.e., as a limit of Riemann–Stieltjes sums. This is natural, since in this case one can also define the integrals as *Young integrals* [16].

We will also need the following estimate in order to prove our main theorems.

Theorem 2.1. [11] *Let* $f \in W^\beta_2$ *and* $g \in W^{1-\beta}_1$. *Then we have the estimate*

$$\left| \int_0^t f(s)\, \mathrm{d}g(s) \right| \le \sup_{0 \le s < t \le T} \left| D^{1-\beta}_{t-} g_{t-}(s) \right| \|f\|_{2,\beta}.$$

To conclude the section we also recall *Garsia–Rademich–Rumsey inequality* (see [11] and [10]).

Lemma 2.1. *Let* $p \ge 1$ *and* $\alpha > \frac{1}{p}$. *Then there exists a constant* $C = C(\alpha,p) > 0$ *such that for any continuous function* f *on* $[0,T]$, *and for all* $0 \le s,t \le T$ *we have*

$$|f(t) - f(s)|^p \le C T^{\alpha p - 1} |t-s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}}\, \mathrm{d}x \mathrm{d}y.$$

Föllmer integral. We recall the definition of a forward-type Riemann–Stieltjes integral due to Föllmer [9] (see [14] for English translation) and discuss its connection to the generalized Lebesgue–Stieltjes integral of Proposition 2.1.

Definition 2.3. Let $(\pi_n)_{n=1}^{\infty}$ be a sequence of partitions $\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}$ such that $|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let X be a continuous process. The *Föllmer integral along the sequence* $(\pi_n)_{n=1}^{\infty}$ of Y with respect to X is defined as

$$\int_0^t Y_u \,\mathrm{d}X_u = \lim_{n\to\infty} \sum_{t_j^n \in \pi_n \cap (0,t]} Y_{t_{j-1}^n}(X_{t_j^n} - X_{t_{j-1}^n}),$$

if the limit exists almost surely.

Remark 2.3. (i) The Föllmer integral is a *forward-type* Riemann–Stieltjes integral. Thus, if the Riemann–Stieltjes integral exists, so does the Föllmer integral, but not necessarily vice versa. Also, it is clear that the Föllmer integral is a pathwise generalization of the Itô integral, if one takes the sequence of partitions $(\pi_n)_{n=1}^{\infty}$ to be refining.

(ii) The Föllmer integral is the natural notion of integration in mathematical finance. Indeed, the budget constraint of a self-financing trading strategy translates in the limit as a Föllmer integral. See [4, 5] for further discussion.

In general, it is very difficult to prove the existence of Föllmer integral. In the case of so-called quadratic variation processes the existence is guaranteed by the Itô–Föllmer formula of Lemma 2.2 below, which shows that the Föllmer integral behaves like the Itô integral in the case of integrators with quadratic variation.

Definition 2.4. Let $(\pi_n)_{n=1}^{\infty}$ be a sequence of partitions $\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}$ such that $|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0$ as $n \to \infty$. Let X be a continuous process. Then X is a *quadratic variation process along the sequence* $(\pi_n)_{n=1}^{\infty}$ if the limit

$$\langle X\rangle_t = \lim_{n\to\infty} \sum_{t_j^n \in \pi_n \cap (0,t]} \left(X_{t_j^n} - X_{t_{j-1}^n}\right)^2$$

exists almost surely

Remark 2.4. (i) For standard Brownian motion W we have $\mathrm{d}\langle W\rangle_t = \mathrm{d}t$ if the sequence (π_n) is refining. This follows from the Borel-Cantelli lemma.

(ii) For continuous martingales M their bracket $\langle M\rangle$ is also their quadratic variation (in the pathwise, Föllmer, sense) for suitably chosen sequences (π_n).

(iii) If Z is a continuous process with zero quadratic variation along (π_n) and X is a continuous quadratic variation process along (π_n) then $\mathrm{d}\langle X + Z\rangle_t = \mathrm{d}\langle X\rangle_t$. This follows from the Cauchy-Schwartz inequality.

(iv) If X is a quadratic variation process along (π_n) and $f \in C^1$ then $Y = f \circ X$ is also a quadratic variation process along (π_n). Indeed,

$$\mathrm{d}\langle Y\rangle_t = f'(X_t)\,\mathrm{d}\langle X\rangle_t.$$

Lemma 2.2. [9] *Let X be a continuous quadratic variation process and let $f \in C^{1,2}([0,T]\times\mathbb{R})$. Let $0 \le s < t \le T$. Then*

$$\begin{aligned} f(t,X_t) &= f(s,X_s) + \int_s^t \frac{\partial f}{\partial t}(u,X_u)\,\mathrm{d}u + \int_s^t \frac{\partial f}{\partial x}(u,X_u)\,\mathrm{d}X_u \\ &\quad + \frac{1}{2}\int_s^t \frac{\partial^2 f}{\partial x^2}(u,X_u)\,\mathrm{d}\langle X\rangle_u. \end{aligned}$$

In particular, the Föllmer integral exists and has a continuous modification.

3. Main Results

Notations, Definitions and Auxiliary Results.

Definition 3.1. Let X be a centered Gaussian process. We denote by $R(t,s)$, $W(t,s)$, and $V(t)$ its covariance, incremental variance and variance, i.e.

$$\begin{aligned} R(t,s) &= \mathbb{E}[X_tX_s], \\ W(t,s) &= \mathbb{E}[(X_t - X_s)^2], \\ V(t) &= \mathbb{E}[X_t^2]. \end{aligned}$$

We denote by $w^*(t)$ the "worst case" incremental variance

$$w^*(t) = \sup_{0\le s\le T-t} W(t+s,s).$$

We begin with the following technical Lemma.

Lemma 3.1. *Let R be a covariance of a centered process with $R(s,t) > 0$. Let $0 < s \le t \le T$.*

(i) *If $R(s,s) \le R(s,t)$, then*

$$1 - \frac{R(s,s)}{R(t,s)} \le \frac{\sqrt{W(t,s)}}{\sqrt{V(s)}},$$

(ii) *if $R(s,s) > R(s,t)$, then*

$$\frac{R(s,s)}{R(t,s)} - 1 \le \frac{\sqrt{W(t,s)}}{\sqrt{V(s)}}\frac{R(s,s)}{R(t,s)}.$$

Proof. For the claim (i), note that we have always $R(t,s)^2 \le R(t,t)R(s,s)$. Hence

$$\frac{R(t,s)^2}{R(s,s)} + R(s,s) - 2R(t,s) \le R(t,t) + R(s,s) - 2R(t,s).$$

Taking square root we obtain

$$\frac{R(t,s)}{\sqrt{R(s,s)}} - \sqrt{R(s,s)} \le \sqrt{W(t,s)}.$$

It remains to note that

$$1-\frac{R(s,s)}{R(t,s)} \leq \frac{R(t,s)}{R(s,s)}-1.$$

For the claim (ii), arguing as in the the proof of the claim (i) above, we obtain that

$$1-\frac{R(t,s)}{R(s,s)} \leq \frac{\sqrt{W(t,s)}}{\sqrt{V(s)}}.$$

Multiplying by $\frac{R(s,s)}{R(t,s)}$ gives the result. □

The key lemma to our analysis is the following estimate for the probability that the process X crosses a fixed level: $\mathbb{P}(X_s < a < X_t)$. Depending on the values of t, s and a one can obtain different estimates and detailed bounds that are given in Lemma A.1 in the appendix. For our purposes we need the following universal estimate which holds for every value s, t and a.

Lemma 3.2. *Let X be a centered Gaussian process with strictly positive and bounded covariance function R, $0 < s < t \leq T$ and $a \in \mathbb{R}$. Then there exists a universal constant C such that*

$$\begin{aligned}&\mathbb{P}\big(X_s < a < X_t\big)\\ \leq\ & C\frac{\sqrt{W(t,s)}}{\sqrt{V(s)}}\left[1+\frac{R(s,s)}{R(t,s)}+\frac{|a|e^{-\frac{a^2}{2V^*}}}{\sqrt{V(s)}}\max\left(1,\frac{R(s,s)}{R(t,s)}\right)\right],\end{aligned}$$

where

$$V^* = \sup_{s\leq T} V(s).$$

Proof. The claim follows directly by applying Lemma 3.1 and the inequality $\sigma^2 \leq W(t,s)$ on the terms in Lemma A.1. □

Recall that a process $X = (X_t)_{t\in[0,T]}$ is *Hölder continuous of order α* if there exists almost surely finite random variable C_T such that

$$|X_t - X_s| \leq C_T|t-s|^\alpha$$

almost surely for all $s,t \in [0,T]$.

Definition 3.2. A centered continuous Gaussian process $X = (X_t)_{t\in[0,T]}$ with covariance R belongs to the *class $\mathcal{X}^\alpha$* if

(i) $R(s,t) > 0$ for every $s,t > 0$,
(ii) the "worst case" incremental variance satisfies, at $t = 0$,

$$w^*(t) = Ct^{2\alpha} + o(t^{2\alpha}),$$

where $C > 0$ and $0 < \alpha < 1$,
(iii) there exists a $c, \delta > 0$ such that

$$V(s) \geq cs^2,$$

when $s \leq \delta$,

(iv) there exists a $\delta > 0$ such that

$$\sup_{0<t<2\delta} \sup_{\frac{t}{2}\le s\le t} \frac{R(s,s)}{R(t,s)} < \infty.$$

Definition 3.2 of the class $\mathcal{X}^\alpha$ is rather technical. However, the next remarks and examples should convince the reader that the assumptions are relatively natural and that the class $\mathcal{X}^\alpha$ quite large.

Remark 3.1. (i) The first condition on strictly positive autocorrelation $R(t,s) > 0$ is rather restrictive. However, it seems that most Gaussian models do indeed satisfy it. Also, if one uses the generalized Lebesgue–Stieltjes approach, one has to assume $R(s,t) > 0$. Indeed, otherwise our integrands would not belong to the required fractional Besov spaces.

(ii) The second condition on the "worst case" incremental variance is the most important assumption: it implies that X has a version which is Hölder continuous of order r for any $r < \alpha$ on $[0,T]$. Indeed, now

$$\mathbb{E}\left[(X_t - X_s)^p\right] \le C_p^p |t-s|^{p\alpha}$$

for every $p \ge 1$. Hence the Hölder continuity follows directly from the Kolmogorov continuity theorem.

(iii) The third condition implies that the process is not too smooth. Indeed, if the variance $V(s)$ behaves like s^γ for some $\gamma \ge 2$ near zero, we obtain that the process is differentiable in the mean square sense. As a consequence we could apply standard integration techniques for such cases.

(iv) Finally, the fourth assumption is quite mild as for it we simply need that when s and t are both close to each other and at the same time near to zero, the variance $V(s)$ is not "too far" from the covariance $R(s,t)$. The fourth condition is also connected to the notion of *local non-determinism* introduced by Berman [6] in connection to the existence of local time as an occupation density.

Example 3.1. Let $0 \le s \le t \le T$.

(i) For processes with stationary increments

$$\begin{aligned} R(t,s) &= \frac{1}{2}\big[V(t) + V(s) - V(t-s)\big], \\ W(t,s) &= V(t-s), \\ w^*(t) &= V(t). \end{aligned}$$

Consequently, a process with stationary increments belongs to $\mathcal{X}^\alpha$ if and only if $V(t) > 0$ for all $t > 0$ and $V(t) = Ct^{2\alpha} + o(t^{2\alpha})$ at $t = 0$.

In particular, the fractional Brownian motion with index $\alpha \in (0,1)$ belongs to the space $\mathcal{X}^\alpha$.

(ii) For stationary processes

$$\begin{aligned} R(t,s) &= r(t-s), \\ W(t,s) &= 2\big[r(0)-r(t-s)\big], \\ V(t) &= r(0), \\ w^*(t) &= 2\big[r(0)-r(t)\big]. \end{aligned}$$

Consequently, a stationary process belongs to the class $\mathcal{X}^\alpha$ if and only if $r(t)>0$ for all t and at $t=0$ we have

$$r(0)-r(t) = Ct^{2\alpha} + o(t^{2\alpha}).$$

In particular, the fractional Ornstein–Uhlenbeck processes with index $\alpha\in(0,1)$ belongs to the space $\mathcal{X}^\alpha$.

Existence of Generalized Lebesgue–Stieltjes Integral. We consider existence of pathwise integrals of the type

$$\int_0^T f'_-(X_u)\,\mathrm{d}Y_u, \tag{3.1}$$

where X and Y are Gaussian processes and f'_- is left derivative of a convex function. We begin by showing that the integral can be understood in generalized Lebesgue–Stieltjes sense, and then we prove that it can also be understood as a forward type integral in the sense of Föllmer. The reason for considering a mixed type integral with two processes X and Y is that we want to later consider mixed processes $Y = M + X$, where M is a Gaussian martingale with Lipschitz-variance and X is a Gaussian process from the class $\mathcal{X}^\alpha$.

Theorem 3.1. *Let $X\in\mathcal{X}^\alpha$. If $\alpha>\beta$, then for every convex function f we have $f'_-(X_\cdot)\in W_2^\beta$ almost surely.*

Proof. Let us first note that it is sufficient to consider convex functions of the form $f(x) = (x-a)^+$, $a\in\mathbb{R}$. Indeed, assume for a moment that the result is valid for these particular convex functions and let f be a general convex function for which the measure f'' compact support. Then

$$|f'_-(X_t)-f'_-(X_s)| \le \int_{\mathrm{supp}(f'')} \mathbf{1}_{X_t<a<X_s} + \mathbf{1}_{X_s<a<X_t}\, f''(\mathrm{d}a).$$

Now, by applying Tonelli's theorem, and the result for the functions $(x-a)^+$, $a\in\mathbb{R}$, we obtain the result for the convex functions f with $\mathrm{supp}(f'')$ compact. Finally, the general case follows by approximating a convex function f with a sequence f_n of convex functions for which f''_n has compact support (see [2] or [3] for details).

Consider then the functions $f(x) = (x-a)^+$, $a\in\mathbb{R}$. Now,

$$|f'_-(X_t)-f'_-(X_s)| = \mathbf{1}_{X_t<a<X_s} + \mathbf{1}_{X_s<a<X_t}.$$

For the first term in the fractional Besov norm we obtain

$$\int_0^T \frac{\mathbf{1}_{X_t>a}}{t^\beta}\,\mathrm{d}t \le \int_0^T \frac{1}{t^\beta}\,\mathrm{d}t < \infty.$$

It follows that we only have to prove that

$$I=\int_0^T\int_0^t \frac{\mathbf{1}_{X_s<a<X_t}+\mathbf{1}_{X_t<a<X_s}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t<\infty.$$

Now, the iterated integral above can be split as

$$\begin{aligned} I &= \int_0^{2\delta}\int_{\frac{t}{2}}^t \frac{\mathbf{1}_{X_s<a<X_t}+\mathbf{1}_{X_t<a<X_s}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t \\ &\quad+\int_{2\delta}^T\int_{t-\delta}^t \frac{\mathbf{1}_{X_s<a<X_t}+\mathbf{1}_{X_t<a<X_s}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t \\ &\quad+\int_{2\delta}^T\int_0^{t-\delta} \frac{\mathbf{1}_{X_s<a<X_t}+\mathbf{1}_{X_t<a<X_s}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t \\ &\quad+\int_0^{2\delta}\int_0^{\frac{t}{2}} \frac{\mathbf{1}_{X_s<a<X_t}+\mathbf{1}_{X_t<a<X_s}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t \\ &= I_1+I_2+I_3+I_4. \end{aligned}$$

In the integrals I_3 and I_4 we can bound indicators with one and they are still finite. Hence, it is sufficient to consider the integrals I_1 and I_2. First, we note that by taking expectation and applying Tonelli's theorem we have the result if

$$\int_{2\delta}^T\int_{t-\delta}^t \frac{\mathbb{P}(X_t<a<X_s)+\mathbb{P}(X_t>a>X_s)}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t<\infty, \tag{3.2}$$

and

$$\int_0^{2\delta}\int_{\frac{t}{2}}^t \frac{\mathbb{P}(X_t<a<X_s)+\mathbb{P}(X_t>a>X_s)}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t<\infty. \tag{3.3}$$

We begin with the term (3.2). By symmetry we can only analyze the term $\mathbb{P}(X_t>a>X_s)$. We have

$$W(t,s)\le w^*(t-s)\le \sup_{t\in[0,T]} V(t)=V^*.$$

Now, since X is continuous on $[0,T]$, $V^*<\infty$. Hence, by assumption (iii) of Definition 3.2 and Lemma 3.2, it is sufficient to consider integral of form

$$\int_{2\delta}^T\int_{t-\delta}^t \frac{\sqrt{W(t,s)}}{(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t.$$

Let now δ be small enough such that, by assumption (ii) of Definition 3.2, we have

$$w^*(t)\le Ct^{2\alpha}$$

for some constant C. Since $\alpha>\beta$ we obtain that (3.2) holds almost surely. Consider next term in (3.3). Note first that by applying Lemma 3.2 together with (iii) and (iv) of Definition 3.2 we obtain

$$\int_0^{2\delta}\int_{\frac{t}{2}}^t \frac{(t-s)^\alpha}{s(t-s)^{\beta+1}}\,\mathrm{d}s\mathrm{d}t\ \le\ C\int_0^{2\delta} t^{-1}t^{\alpha-\beta}\,\mathrm{d}t\ <\ \infty.$$

Hence it is sufficient to study term of form

$$e^{-\frac{a^2}{2V^*}}\frac{|a|\sqrt{W(t,s)}}{V(s)}.$$

The case $a = 0$ is trivial. Let now $a \neq 0$ and introduce time points $t_0 = 2\delta$,

$$t_k = 2\delta - 2\delta \sum_{j=1}^{k} \left(\frac{1}{2}\right)^j, \quad k \geq 1.$$

Now the integral can be spit as

$$\int_0^{2\delta} \int_{\frac{t}{2}}^{t} \ldots \mathrm{d}s\mathrm{d}t = \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{\frac{t}{2}}^{t} \ldots \mathrm{d}s\mathrm{d}t.$$

Moreover,

$$\sup_{t_{k+1} \leq s \leq t_k} V(s) \leq C t_k^{2\alpha}.$$

Hence by applying Lemma 3.2 we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{\frac{t}{2}}^{t} \frac{\mathbb{P}(X_t > a > X_s)}{(t-s)^{\beta+1}} \mathrm{d}s\mathrm{d}t &\leq C \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{\frac{t}{2}}^{t} \frac{e^{-\frac{a^2}{2Ct_k^{2\alpha}}}(t-s)^{\alpha}}{s^2(t-s)^{\beta+1}} \mathrm{d}s\mathrm{d}t \\
&\leq C \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} e^{-\frac{a^2}{2Ct_k^{2\alpha}}} t^{-2} t^{\alpha-\beta} \mathrm{d}t \\
&\leq C \sum_{k=0}^{\infty} e^{-\frac{a^2}{2Ct_k^{2\alpha}}} t_k^{\alpha-\beta-1} \\
&< \infty.
\end{aligned}$$

Hence the result is valid for convex function $f(x) = (x-a)^+$ and the claim follows. □

Corollary 3.1. *Let f be a linear combination of convex functions, $Y \in W_1^{1-\beta}$ and $X \in \mathcal{X}^{\alpha}$. If $\alpha > \beta$, then the integral*

$$\int_0^T f'_-(X_u)\, \mathrm{d}Y_u$$

exists almost surely in the sense of generalized Lebesgue-Stieltjes integral.

Corollary 3.2. *Let f be a linear combination of convex functions, $Y \in W_1^{1-\beta}$ and $X \in \mathcal{X}^{\alpha}$. Put $S = g(X)$ for some strictly monotone function $g \in C^1$. If $\alpha > \beta$, then the integral*

$$\int_0^T f'_-(S_u) g'(X_u)\, \mathrm{d}Y_u$$

exists almost surely in the sense of generalized Lebesgue-Stieltjes integral.

Proof. Since g is strictly monotone, the inverse g^{-1} exists and we have

$$\mathbf{1}_{S_t > a > S_s} = \mathbf{1}_{X_t > g^{-1}(a) > X_s}.$$

Hence, by following the proof in [2] we obtain the result from our main theorem if

$$\int_0^T \int_0^t \frac{|f'_-(S_t)||g'(X_t) - g'(X_s)|}{(t-s)^{\beta+1}} \mathrm{d}s\, \mathrm{d}t < \infty.$$

We obtain by Taylor's theorem that

$$|g'(X_t) - g'(X_s)| = |g'(\xi)||X_t - X_s|$$

for some random point ξ between X_s and X_t. It remains to note that

$$\mathbb{E}|X_t - X_s| \leq \sqrt{\mathbb{E}|X_t - X_s|^2} = \sqrt{W(t,s)},$$

and the claim follows. □

Remark 3.2. For financial applications natural candidate is $g(x) = e^x$.

Next our aim is to define integrals over the random interval $[0, \tau]$, where τ is almost surely finite random variable instead of deterministic time.

Theorem 3.2. *Let $X_t \in \mathcal{X}^\alpha$ and let $\tau \leq T$ be a random time. If $\alpha > \beta$, then for every convex function f we have $f'_-(X_\cdot)\mathbf{1}_{\cdot\leq\tau} \in W_2^\beta$ almost surely.*

Proof. For the first term in the fractional Besov norm we can simply make upper bound

$$\mathbf{1}_{t\leq\tau} \leq 1$$

and we can proceed as in the proof of our main theorem 3.1.

Consider then the second term in the fractional Besov norm:

$$\int_0^T \int_0^t \frac{|f'_-(X_t)\mathbf{1}_{t\leq\tau} - f'_-(X_s)\mathbf{1}_{s\leq\tau}|}{(t-s)^{\beta+1}} \mathrm{d}s\mathrm{d}t.$$

Now either $s \leq t \leq \tau$ in which case we can proceed as for deterministic time or we have $t > \tau$ and $s < \tau$. In this case we get

$$\begin{aligned} \int_\tau^T \int_0^\tau \frac{|f'_-(X_s)|}{(t-s)^{\beta+1}} \mathrm{d}s\mathrm{d}t &\leq C(\beta) \sup_{s\in[0,T]} |f'_-(X_s)| \int_\tau^T (u-\tau)^{-\beta} \,\mathrm{d}u \\ &\leq C(\beta) \sup_{s\in[0,T]} |f'_-(X_s)| T^{1-\beta}. \end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let f be a linear combination of convex functions, $Y \in W_1^{1-\beta}$, $X \in \mathcal{X}^\alpha$, and let $\tau \leq T$ be a bounded random time. If $\alpha > \beta$, then the integral*

$$\int_0^\tau f'_-(X_u)\,\mathrm{d}Y_u$$

exists almost surely in the sense of generalized Lebesgue–Stieltjes integral.

Remark 3.3. Let g be strictly monotone, and set $S = g(X)$. Then similar results hold for integrals

$$\int_0^\tau f'_-(S_u) g'(X_u)\,\mathrm{d}Y_u.$$

Existence of Föllmer Integral. As noted e.g. by Zähle [17] we can sometimes approximate the generalized Lebesgue–Stieltjes integral with Riemann–Stieltjes sums. This is the topic of Theorem 3.3 below. The proof follows exactly the same arguments as for the particular case for fractional Brownian motion in [2]. Hence we only give the idea of the proof and details are left to the reader.

Theorem 3.3. *Let f be a linear combination of convex functions, $Y \in W_1^{1-\beta}$ and $X \in \mathcal{X}^\alpha$. If $\alpha > \beta$, then*

$$\sum_{j=1}^{k(n)} f'_-(X_{u_j^n})(Y_{u_{j+1}^n} - Y_{u_j^n}) \to \int_0^T f'_-(X_u)\mathrm{d}Y_u$$

almost surely for any partition $\pi_n = \{0 = u_0^n < \ldots < u_{k(n)}^n = T\}$ such that $|\pi_n| \to 0$.

Proof. Again we can assume that the measure f'' has compact support. Now,

$$\int_0^T f'_-(X_u)\,\mathrm{d}Y_u - \sum_{j=1}^{k(n)} f'_-(X_{u_j^n})(Y_{u_{j+1}^n} - Y_{u_j^n}) = \int_0^T h_n(X_u)\mathrm{d}Y_u,$$

where

$$h_n(u) = \sum_{j=1}^{k(n)} \left(f'_-(X_{u_j^n}) - f'_-(X_u)\right) \mathbf{1}_{u_j^n < u \le u_{j+1}^n}.$$

To conclude the proof we have to show that

$$\|h_n\|_{\beta,2} \to 0$$

almost surely. Following arguments in [2] we obtain an integrable upper bound in both terms of the fractional Besov norm, and hence the result follows by using the dominated convergence theorem. More precisely, we have

$$|h_n(t)| \le 2 \sup_{0\le u\le T} |f'_-(X_u)|,$$

and

$$|h_n(t) - h_n(s)| \le C \int \mathbf{1}_{X_s < a < X_t} f''(\mathrm{d}a)$$

on set $\{X_s < a < X_t\}$ and

$$|h_n(t) - h_n(s)| \le C \int \mathbf{1}_{X_t < a < X_s} f''(\mathrm{d}a)$$

on set $\{X_t < a < X_s\}$. Hence we have integrable upper bound in all the cases and hence the result follows by dominated convergence theorem together with the fact that

$$|h_n(t)| \to 0$$

almost surely. □

Remark 3.4. Let g be strictly monotone and set $S = g(X)$. Let $\tau \le T$ be a random time. Then similar result holds for integrals of the form

$$\int_0^\tau f'_-(S_u) g'(X_u)\,\mathrm{d}Y_u.$$

Existence of Mixed Integrals. The particular reason why we considered integrals of form (3.1) with arbitrary process Y instead of X is that now we can apply our result for processes of type

$$Y = M + X$$

where M is a centered Gaussian martingale with Lipschitz continuous variance function $\langle M \rangle$. These kind of mixed processes are especially interesting in mathematical finance, see [4, 5]

Note that Lipschitz continuity of the variance on M implies that

$$\mathbb{E}[w_M^*(t)] \leq Ct$$

Theorem 3.4. *Let M be centered Gaussian martingale with Lipschitz variance and let X be a Gaussian process from the class $\mathcal{X}^\alpha$ for some $\alpha > 1/2$. Denote $Y = M + X$. Let f be a linear combination of convex functions and let $\tau \leq T$ be a stopping time. Then the integral*

$$\int_0^\tau f'_-(Y_u)\,\mathrm{d}Y_u = \int_0^\tau f'_-(Y_u)\,\mathrm{d}M_u + \int_0^\tau f'_-(Y_u)\,\mathrm{d}X_u \tag{3.4}$$

exists as a Föllmer integral.

Remark 3.5. Note that we do not assume that the processes M and X are independent.

Proof. Consider first the first integral on the right-hand side of (3.4). Since M is a martingale this integral exists as Itô integral. By using suitable subdivision sequence $(\pi_n)_{n=0}^\infty$ it exists pathwise as a Föllmer integral.

Consider then the second integral on the right-hand side of (3.4). Since M has Lipschitz variance we have that $Y \in \mathcal{X}^{1/2}$. Consequently, $f'_-(Y)\mathbf{1}_{\cdot \leq \tau} \in W_2^\beta$ for all $\beta \in (1-\alpha, 1/2)$. Therefore, the generalized Lebesgue–Stieltjes integral $\int_0^\tau f'_-(Y_u)\,\mathrm{d}X_u$ exists.

Finally, all the integrals in (3.4) can also be understood as Föllmer integrals due to Theorem 3.3. □

Itô–Tanaka Formula. We begin with the following Itô formula for smooth functions. The proof is based on Taylor expansion and is the same as in the semimartingale case, or indeed, in the classical case.

Proposition 3.1. *Let $X \in \mathcal{X}^\alpha$ with $\alpha > \frac{1}{2}$ and let $f \in C^2$. Then*

$$f(X_T) = f(X_0) + \int_0^T f'(X_u)\,\mathrm{d}X_u$$

The Itô formula of Proposition 3.1 can be extended to convex functions. Indeed, the arguments presented in [2] imply that:

Theorem 3.5. *Let $X \in \mathcal{X}^\alpha$ with $\alpha > \frac{1}{2}$ and let f be a convex function. Then*

$$f(X_T) = f(X_0) + \int_0^T f'_-(X_u)\,\mathrm{d}X_u.$$

Corollary 3.4. *Let g be a strictly monotone smooth function and set $S = g(X)$ for some $X \in \mathcal{X}^\alpha$. Let f be a linear combination of convex functions and let $\tau \le T$ be a random time. Then*

$$f(S_\tau) = f(S_0) + \int_0^\tau f'_-(S_u)\,\mathrm{d}S_u.$$

Let us now consider the non-smooth case.

Föllmer [9] showed that for any process Y with finite quadratic variation $\langle Y\rangle$ and $f \in C^2$ we have

$$f(Y_T) = f(Y_0) + \int_0^T f'(Y_u)\,\mathrm{d}Y_u + \frac{1}{2}\int_0^T f''(Y_u)\,\mathrm{d}\langle Y\rangle_u,$$

which also implies the existence of the Föllmer integral $\int_0^T f'(Y_u)\,\mathrm{d}Y_u$. We will extend this result to convex functions f. We will, however, not consider general quadratic variation processes. Instead, we consider processes that are of the form $Y = M + X$ as in Theorem 3.4.

The non-trivial quadratic variation gives rise to local time:

Definition 3.3. Let Y be a continuous process with quadratic variation $\langle Y\rangle$. Its *local time* $L^a_T(Y)$ is the process defined by the *occupation time formula*

$$\int_{-\infty}^{\infty} g(a)L^a_T(Y)\,\mathrm{d}a = \int_0^T g(Y_u)\,\mathrm{d}\langle Y\rangle_u \tag{3.5}$$

almost surely for every bounded, Borel measurable function g.

Theorem 3.6. *Let f and $Y = M + X$ satisfy the conditions of Theorem 3.4. Then there exists a local time process $L^a_T(Y)$ such that*

$$f(Y_T) = f(Y_0) + \int_0^T f'_-(Y_u)\,\mathrm{d}Y_u + \frac{1}{2}\int_{-\infty}^{\infty} L^a_T(Y)\,f''(\mathrm{d}a). \tag{3.6}$$

Proof. Note first that Theorem 3.4 implies the existence of the integrals as mixed Föllmer and generalized Lebesgue–Stieltjes integral. Let us consider these integrals for a while.

First by Taylor expansion we have the result for $f \in C^2$. The general case follows with same arguments as for semimartingale case (see [12], pp. 221–224). Indeed, let f be a convex function for which the measure f'' has compact support and define

$$f_n(x) = n\int_{-\infty}^0 f(x+y)j(ny)\,\mathrm{d}y,$$

where $j(y)$ is a positive C^∞-function with compact support in $(-\infty, 0]$ such that

$$\int_{-\infty}^0 j(y)\,\mathrm{d}y = 1.$$

Now f_n is well-defined and smooth for every n. Moreover, f_n converges to f pointwise and $(f_n)'_-$ increases to f'_-. Now,

$$f_n(Y_T) = f_n(Y_0) + \int_0^T (f_n)'_-(Y_u)\,\mathrm{d}Y_u + \frac{1}{2}\int_0^T f''_n(Y_u)\,\mathrm{d}\langle M\rangle_u.$$

Moreover, by examining the proof of Theorem 3.5 we can conclude that

$$\int_0^T (f_n)'_-(Y_u)\mathrm{d}X_u \to \int_0^T f'_-(Y_u)\,\mathrm{d}X_u,$$

and for the Itô integral we get by standard arguments that

$$\int_0^T (f_n)'_-(Y_u)\,\mathrm{d}M_u \to \int_0^T f'_-(Y_u)\,\mathrm{d}M_u.$$

Consequently, we obtain that for every convex function f there exists a process A_f such that

$$f(Y_T) = f(Y_0) + \int_0^T f'_-(Y_u)\,\mathrm{d}Y_u + \frac{1}{2}A_f^T.$$

Applying this result to the convex functions $(x-a)^+$ and $(x-a)^-$ we obtain

$$(Y_T - a)^+ = (Y_0 - a)^+ + \int_0^T \mathbf{1}_{Y_u > a}\,\mathrm{d}Y_u + \frac{1}{2}A_+^T$$

and

$$(Y_T - a)^- = (Y_0 - a)^- + \int_0^T \mathbf{1}_{Y_u \leq a}\,\mathrm{d}Y_u + \frac{1}{2}A_-^T.$$

Subtract second equation from the first to get that

$$A_+^T = A_-^T$$

almost surely, and we define the local time process $L_T^a(Y) = A_+^T$. Moreover, adding second equation to the first we get

$$|Y_T - a| = |Y_0 - a| + \int_0^T \operatorname{sgn}(Y_u - a)\,\mathrm{d}Y_u + L_T^a(Y).$$

In order to have (3.6) for convex function f for which f'' has compact support we use representations

$$f(x) = \alpha x + \beta + \int_{-\infty}^{\infty} |x-a|\, f''(\mathrm{d}a)$$

and

$$f'_-(x) = \alpha + \int_{-\infty}^{\infty} \operatorname{sgn}(x-a)\, f''(\mathrm{d}a).$$

By using the representation formula for the convex function $|Y_T - a|$ we obtain

$$\begin{aligned} f(Y_T) &= f(Y_0) + \alpha(Y_T - Y_0) \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2}\left(\int_0^T \operatorname{sgn}(Y_u - a)\,\mathrm{d}Y_u + L_T^a(Y)\right) f''(\mathrm{d}a). \end{aligned}$$

Hence, for convex functions f for which f'' has compact support, the process A_f is given by

$$\begin{aligned} A_f^T &= \int_{-\infty}^{\infty} L_T^a(Y)\, f''(\mathrm{d}a) + \int_{-\infty}^{\infty}\int_0^T \operatorname{sgn}(Y_u - a)\,\mathrm{d}Y_u\, f''(\mathrm{d}a) \\ &\quad - \int_0^T f'_-(Y_u)\,\mathrm{d}Y_u + \alpha(Y_T - Y_0). \end{aligned}$$

It remains to apply Fubini's Theorem to stochastic integrals. For the martingale part we use classical Stochastic Fubini's Theorem (see [12]) and for pathwise integral we can use classical Fubini's Theorem for σ-finite measures. Now, we obtain the occupation time formula by the same argument as in the semimartingale case. □

Corollary 3.5. *Let M and X be processes such that assumptions of Theorem 3.4 are satisfied. Then the local time $L_T^a(Y)$ of the process $Y = X+M$ is almost surely continuous in t and in a and we have the* local representation

$$L_T^a(Y) = \lim_{\epsilon\to 0+} \frac{1}{2\epsilon}\int_0^T \mathbf{1}_{a-\epsilon<Y_u<a+\epsilon}\,\mathrm{d}\langle Y\rangle_u$$

almost surely.

Proof. Obviously $(Y_t - a)^+$ is continuous in t and a. Hence it is enough to show that the stochastic integral

$$\int_0^t \mathbf{1}_{Y_u>a}\,\mathrm{d}Y_u$$

is continuous in t and a. For this it is enough to show that

$$\|\mathbf{1}_{X_u>a}\mathbf{1}_{u\le t}\|_{2,\beta}$$

is continuous in t and a. But continuity in t is evident and the continuity in a follows from Lebesgue dominated convergence theorem.

Finally, the local representation follows from continuity of the local time in a and the occupation time formula (3.5). □

Remark 3.6.

(i) In the case of standard Brownian motion W with $g(x) = \mathbf{1}_A$ the occupation time formula reads that

$$\int_A L_T^a(W)\,\mathrm{d}a = \int_0^T \mathbf{1}_{\{W_u\in A\}}\,\mathrm{d}u.$$

So the local time is exactly the density with respect to Lebesgue measure for the occupation measure. In the Itô–Tanaka formula the the local time for Y is the density of the occupation measure with respect to the "clock" $\mathrm{d}\langle Y\rangle$.

(ii) Berman [6] showed that a Gaussian process X admits local time as occupation density with respect to the "Lebesgue clock" $\mathrm{d}u$ if and only if its incremental variance W satisfies

$$\int_0^T\int_0^T \frac{1}{\sqrt{W(t,s)}}\,\mathrm{d}s\mathrm{d}t < \infty.$$

Consequently, every process $X \in \mathcal{X}^\alpha$ admits a local time defined as the density of the occupation measure with respect to the "Lebesgue clock". If $\alpha > 1/2$, then $L_T^a(X) = 0$.

(iii) In Corollary 3.5 we have

$$\mathrm{d}\langle Y\rangle_u = \mathrm{d}\langle M\rangle_u = \big(\langle M\rangle'_-\big)_u\,\mathrm{d}u.$$

(iv) Again with similar arguments we can obtain the Itô–Tanaka formula for transformation $S = g(Y)$ with obvious changes.

4. Implications to Option-Pricing

We explain the implications of Itô–Tanaka formula for pricing and hedging of European options in the spirit on Sondermann [14, Ch. 6]. For the use of non-semimartingales and pathwise Föllmer integration in mathematical finance we refer to [5].

Let S be the discounted stock-price process given by the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \mu(t)\mathrm{d}t + \mathrm{d}Y_t,$$

where Y is a centered continuous quadratic variation process. Then, by [5],

$$S_t = S_0 \exp\left\{ \int_0^t \mu(u)\,\mathrm{d}u + Y_t - \frac{1}{2}\langle Y\rangle_t \right\}.$$

Suppose we want to replicate the European call-option $(S_T - K)^+$. Suppose $Y \in \mathcal{X}^\alpha$ for some $\alpha > 1/2$. Then $\langle Y\rangle = 0$, and the Itô–Tanaka formula takes the form

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{S_u \geq K}\,\mathrm{d}S_u.$$

This means that one can replicate the European call-option $(S_T - K)^+$ with a fair price $(S_0 - K)^+$ by dynamically buying at time t one share of the stock if the option goes from out-of-the-money into in-the-money and selling the stock if the option goes from out-of-the-money into in-the-money. But this is silly because of at least two reasons:

(i) Take $K > S_0$. Then out-of-the-money options are worthless. This is obviously nonsense!
(ii) Take $K = S_0$. Then one could make profit without risk by selling cheap and buying expensive. This is against any reasonable business sense!

Thus, we see that in order to avoid this *buy–sell paradox* the option-pricing model must include a quadratic variation.

Suppose then that $Y = M + X$ such that the assumption of Theorem 3.6 are satisfied. Then the Itô–Tanaka formula takes the form

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{S_u \geq K}\,\mathrm{d}S_u + \frac{1}{2}L_T^K(S). \tag{4.1}$$

This solves the buy–sell paradox. Indeed, now the buy–sell strategy $\mathbf{1}_{S_u \geq K}$ is no longer a self-financing hedging strategy for the call-option $(S_T - K)^+$.

Remark 4.1. (i) The local time $\frac{1}{2}L_T^K(S)$ in the hedging formula (4.1) can be interpreted as transaction costs in the following way noted by Sondermann [14]: Assume that one tries to apply the buy–sell strategy $\mathbf{1}_{S_u \geq K}$ to hedge the European call-option $(S_T - K)^+$, i.e., buy the stock at price K when up-crossing the barrier K, sell it again when down-crossing the barrier. But you cannot sell it at the same price. You need a so-called *cutout*. You can place only limit orders of the form: buy at K, sell at $K - \epsilon$ for some $\epsilon > 0$. The smaller you choose ϵ, the more cutouts you will face, and in the limit the sum of these cutouts is just equal to the transaction costs.

(ii) The true hedging strategy for the European call-option in the models $Y = M + X$ satisfying the assumptions of Theorem 3.6 can be calculated just like in the martingale case $Y = M$ by using the Black–Scholes BPDE. See [5] for details.

Appendix A. Level-Crossing Lemma

The key lemma in our analysis is the following estimate for the probability that a Gaussian process X crosses a fixed level. Actually, in [3] the authors proved the lemma in the particular case of fractional Brownian motion. We extend the result here for more general Gaussian process. We consider only probability $\mathbb{P}(X_s < a < X_t)$ and a case $\sup_{s\leq T} V(s) \leq 1$. However, by considering processes $Y = -X$ and $Y = \frac{X}{\sqrt{V^*}}$ we obtain same bound for $\mathbb{P}(X_s > a > X_t)$ and for the general case $\sup_{s\leq T} V(s) < \infty$. Also, note that continuous Gaussian processes on compact time intervals satisfy $V^* < \infty$.

Lemma A.1. *Let X be a Gaussian process with positive covariance function R. Denote*

$$\sigma^2 = \frac{R(s,s)R(t,t) - R(t,s)^2}{R(s,s)},$$

and fix $0 < s < t \leq T$ and $a \in \mathbb{R}$. Assume that the variance function satisfies

$$V^* := \sup_{s\leq T} V(s) \leq 1.$$

(i) *If*

$$\frac{R(s,s)}{R(t,s)}(a-1) < a,$$

then there exists a constant C, independent of s, t, T and a, such that

$$\mathbb{P}\big(X_s < a < X_t\big) \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &\leq C\min[\sqrt{V(s)}\sigma, \sigma^2]e^{-\frac{a^2}{2}},\\
I_2 &\leq Ce^{-\frac{\min[a^2,(a-1)^2]}{2V^*}}\frac{\sigma}{\sqrt{V(s)}}\left[\mathbf{1}_{|a|>2} + \left(a - \frac{R(s,s)}{R(t,s)}(a-1)\right)\mathbf{1}_{|a|\leq 2}\right],\\
I_3 &\leq C\frac{R(s,s)}{R(t,s)}\frac{\sigma}{\sqrt{V(s)}}e^{-\frac{\min[a^2,(a-1)^2]}{2V^*}},\\
I_4 &\leq e^{-\frac{a^2}{2V^*}}\frac{1}{\sqrt{V(s)}}\left|a\left(1 - \frac{R(s,s)}{R(t,s)}\right)\right|,
\end{aligned}$$

(ii) *If*

$$\frac{R(s,s)}{R(t,s)}(a-1) \geq a,$$

then there exists a constant C, independent of s, t, T and a, such that

$$\mathbb{P}\big(X_s < a < X_t\big) \leq C\min[\sqrt{V(s)}\sigma, \sigma^2]e^{-\frac{a^2}{2}}.$$

In the proof we use the following well-known estimate.

Lemma A.2. *Let Z be a standard normal random variable and fix $a > 0$. Then*

$$\mathbb{P}\big(Z > a\big) \leq \frac{1}{\sqrt{2\pi}a} e^{-\frac{a^2}{2}}.$$

Proof of Lemma A.1. We make use of decomposition

$$X_t = \frac{R(t,s)}{R(s,s)} X_s + \sigma Y,$$

where Y is $N(0,1)$ random variable independent of X_s and

$$\sigma^2 = \frac{R(t,t)R(s,s) - R(t,s)^2}{R(s,s)}.$$

Assume that

$$\frac{R(s,s)}{R(t,s)}(a-1) < a.$$

Then we obtain

$$\begin{aligned}
&\mathbb{P}(X_s < a < X_t) \\
&= \int_{-\infty}^{a} \mathbb{P}\left(Y \geq \frac{a - \frac{R(t,s)}{R(s,s)}x}{\sigma}\right) \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}}\, \mathrm{d}x \\
&= \int_{-\infty}^{\frac{R(s,s)}{R(t,s)}(a-1)} \mathbb{P}\left(Y \geq \frac{a - \frac{R(t,s)}{R(s,s)}x}{\sigma}\right) \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}}\, \mathrm{d}x \\
&\quad + \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \mathbb{P}\left(Y \geq \frac{a - \frac{R(t,s)}{R(s,s)}x}{\sigma}\right) \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}}\, \mathrm{d}x \\
&= I_1 + A_1.
\end{aligned}$$

Moreover, if

$$\frac{R(s,s)}{R(t,s)}(a-1) \geq a,$$

then

$$\mathbb{P}(X_s < a < X_t) \leq I_1.$$

Note that here I_1 corresponds the one given in the Lemma and A_1 contains I_2, I_3 and I_4. We shall use similar technique for the rest of the proof.

We begin with I_1. By Lemma A.2 we have

$$\mathbb{P}\left(Y \geq \frac{a - \frac{R(t,s)}{R(s,s)}x}{\sigma}\right) \leq \frac{1}{\sqrt{2\pi}A(x)} e^{-\frac{A(x)^2}{2}},$$

where $A(x) = \frac{a - \frac{R(t,s)}{R(s,s)}x}{\sigma}$. Hence

$$\begin{aligned} I_1 &\leq \int_{-\infty}^{\frac{R(s,s)}{R(t,s)}(a-1)} \frac{1}{\sqrt{2\pi}A(x)} e^{-\frac{A(x)^2}{2}} \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}} \, \mathrm{d}x \\ &\leq \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{a^2}{2}} \int_{-\infty}^{\frac{R(s,s)}{R(t,s)}(a-1)} \frac{1}{2\pi} e^{-\frac{A(x)^2}{2} - \frac{x^2}{2V(s)} + \frac{a^2}{2}} \, \mathrm{d}x \\ &= \frac{R(s,s)}{R(t,s)} \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{a^2}{2}} \int_1^\infty \frac{1}{2\pi} e^{-\frac{y^2}{2\sigma^2} - \frac{\left[\frac{R(s,s)}{R(t,s)}(a-y)\right]^2}{2V(s)} + \frac{a^2}{2}} \, \mathrm{d}y \end{aligned}$$

We proceed to study the integral. Now,

$$\begin{aligned} &-\frac{y^2}{2\sigma^2} - \frac{\left[\frac{R(s,s)}{R(t,s)}(a-y)\right]^2}{2V(s)} + \frac{a^2}{2} \\ &= -\frac{1}{2\bar{\sigma}^2}\left[\left(y - a\frac{R(s,s)}{R(t,s)^2}\bar{\sigma}^2\right)^2 + a^2\left(\frac{R(s,s)}{R(t,s)^2}\bar{\sigma}^2 - \bar{\sigma}^2 - \frac{R(s,s)^2}{R(t,s)^4}\bar{\sigma}^4\right)\right], \end{aligned}$$

where

$$\frac{1}{\bar{\sigma}^2} = \frac{1}{\sigma^2} + \frac{R(s,s)}{R(t,s)^2}.$$

Now

$$\frac{1}{\bar{\sigma}^2} \geq 1$$

and since $V^* \leq 1$, we also have

$$\left(\frac{R(s,s)}{R(t,s)^2}\bar{\sigma}^2 - \bar{\sigma}^2 - \frac{R(s,s)^2}{R(t,s)^4}\bar{\sigma}^4\right) \geq 0.$$

Thus,

$$\begin{aligned} &\int_1^\infty \frac{1}{2\pi} e^{-\frac{y^2}{2\sigma^2} - \frac{\left[\frac{R(s,s)}{R(t,s)}(a-y)\right]^2}{2s^{2H}} + \frac{a^2}{2}} \, \mathrm{d}y \\ &\leq \int_1^\infty \frac{1}{2\pi} e^{-\frac{1}{2\bar{\sigma}^2}\left(y - a\frac{R(s,s)}{R(t,s)^2}\bar{\sigma}^2\right)^2} \, \mathrm{d}y \\ &\leq \frac{\bar{\sigma}}{\sqrt{2\pi}}. \end{aligned}$$

Hence, we obtain for I_1 that

$$I_1 \leq C\frac{R(s,s)}{R(t,s)} \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{a^2}{2}} \bar{\sigma}.$$

Now we have

$$\bar{\sigma}^2 = \frac{\sigma^2 R(t,s)^2}{\sigma^2 + R(s,s)},$$

and hence for I_1 there exists a constant C such that

$$I_1 \leq C \min[\sqrt{V(s)}\sigma, \sigma^2] e^{-\frac{a^2}{2}}.$$

For the term A_1 we have

$$\begin{aligned} A_1 &= \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{A(x)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,\mathrm{d}y \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x \\ &= \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{A(x)}^{\frac{1}{\sigma}} \ldots \mathrm{d}y\mathrm{d}x + \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{\frac{1}{\sigma}}^{\infty} \ldots \mathrm{d}y\mathrm{d}x \\ &= A_2 + I_2. \end{aligned}$$

Consider then I_2. Applying Lemma A.2 we obtain

$$I_2 \leq C \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{1}{2\sigma^2}} \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x.$$

Note that $\sigma^2 \geq 0$. Therefore,

$$\frac{R(s,s)}{R(t,s)^2} \geq \frac{1}{R(t,t)} \geq \frac{1}{V^*}.$$

Now if $|a| > 2$, we can apply Lemma A.2 to obtain

$$\int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x \leq e^{-\frac{\min[a^2,(a-1)^2]}{2V^*}}.$$

As a consequence, we obtain the required upper bound for I_2. Now, if $|a| \leq 2$ we obtain

$$\int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x \leq e^{-\frac{\min[a^2,(a-1)^2]}{2V^*}} \left[a - \frac{R(s,s)}{R(t,s)}(a-1)\right].$$

To conclude we study the term A_2. If we have

$$\frac{R(s,s)}{R(t,s)} a < a,$$

then by applying the Tonelli theorem we obtain

$$\begin{aligned} A_2 &= \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{A(x)}^{\frac{1}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,\mathrm{d}y \frac{1}{\sqrt{2\pi}\sqrt{V(s)}} e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x \\ &= \int_{(\left[1-\frac{R(t,s)}{R(s,s)}\right]\frac{a}{\sigma}}^{\frac{1}{\sigma}} \int_{(a-\sigma y)\frac{R(s,s)}{R(t,s)}}^{a} \ldots \mathrm{d}x\mathrm{d}y \\ &= \int_{(\left[1-\frac{R(t,s)}{R(s,s)}\right]\frac{a}{\sigma}}^{\frac{1}{\sigma}} \int_{(a-\sigma y)\frac{R(s,s)}{R(t,s)}}^{\frac{R(s,s)}{R(t,s)}a} \ldots \mathrm{d}x\mathrm{d}y + \int_{(\left[1-\frac{R(t,s)}{R(s,s)}\right]\frac{a}{\sigma}}^{\frac{1}{\sigma}} \int_{\frac{R(s,s)}{R(t,s)}a}^{a} \ldots \mathrm{d}x\mathrm{d}y \\ &= I_3 + I_4. \end{aligned}$$

Moreover, if

$$\frac{R(s,s)}{R(t,s)} a \geq a,$$

then

$$A_2 \leq I_3.$$

Now, for I_3 we have

$$\begin{aligned}&\int_{\left(\left[1-\frac{R(t,s)}{R(s,s)}\right]\frac{a}{\sigma}}^{\frac{1}{\sigma}}\int_{(a-\sigma y)\frac{R(s,s)}{R(t,s)}}^{\frac{R(s,s)}{R(t,s)}a} e^{-\frac{y^2}{2}}\frac{1}{\sqrt{V(s)}}e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x\mathrm{d}y\\ \leq\ & \frac{\sigma}{\sqrt{V(s)}}e^{-\frac{\min[a^2,(a-1)^2]}{2V^*}}\frac{R(s,s)}{R(t,s)}\int_{-\infty}^{\infty}|y|e^{-\frac{y^2}{2}}\,\mathrm{d}y.\end{aligned}$$

Hence, we have the required upper bound for I_3. To conclude, note that for I_4 we have

$$\begin{aligned}I_4 &= \int_{\left(\left[1-\frac{R(t,s)}{R(s,s)}\right]\frac{a}{\sigma}}^{\frac{1}{\sigma}}\int_{\frac{R(s,s)}{R(t,s)}a}^{a} e^{-\frac{y^2}{2}}\frac{1}{\sqrt{V(s)}}e^{-\frac{x^2}{2V(s)}}\,\mathrm{d}x\mathrm{d}y\\ &\leq C\frac{1}{\sqrt{V(s)}}e^{-\frac{a^2}{2V^*}}|a|\left|1-\frac{R(s,s)}{R(t,s)}\right|.\end{aligned}$$

This finishes the proof of Lemma A.1. □

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Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FIN-65101 Vaasa, FINLAND

Department of Mathematics and System Analysis, Aalto University School of Science, Helsinki, P.O. Box 11100, FIN-00076 Aalto, FINLAND