NO-ARBITRAGE WITH NON-SEMIMARTINGALES

CONTINUOUS SIMPLE ARBITRAGE CASE

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We discuss some recent notions and results connected to simple arbitrage with general continuous pricing models. The discussion is based on the following works (and other works):

Outline

1 Simple Arbitrage

2 Conditional Full Support

3 Delay-Simple Arbitrage
Outline

1. Simple Arbitrage

2. Conditional Full Support

3. Delay-Simple Arbitrage
$S = (S_t)_{t \in [0,T]}$ is a continuous \textbf{DISCOUNTED STOCK-PRICE PROCESS} on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$.

**Definition (\(\mathcal{T}\)-Simple Strategy)**

A \textbf{\(\mathcal{T}\)-simple strategy} is a trading strategy where the number of stocks owned by the investor at time $t$ is

$$\Phi_t = \varphi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}.$$

\(\varphi_j\) are $\mathcal{F}_{\tau_j}$-measurable, $n \in \mathbb{N}$ is fixed, $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ are stopping times, satisfying some additional condition $\mathcal{T}$ to be specified later. If there are no additional conditions, we omit the $\mathcal{T}$. 

**Simple Arbitrage**

(0-Admissible) $\mathcal{T}$-Simple Arbitrage

The **wealth process** associated with strategy $\Phi$ with initial endowment $v$ is

$$V_t(\Phi, v) = v + \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (S_{t \wedge \tau_{j+1}} - S_{t \wedge \tau_j}).$$

**Definition ((0-Admissible) $\mathcal{T}$-Simple Arbitrage)**

A $\mathcal{T}$-Simple strategy is

- a **$\mathcal{T}$-Simple Arbitrage** if $V_T(\Phi, 0) \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[V_T(\Phi, 0) > 0] > 0$.
- **c-Admissible** if $V_t(\Phi, 0) \geq -c$ for all $t \in [0, T]$ $\mathbb{P}$-a.s.
**Simple Arbitrage**
*Conditional Up’n’Down (CUD)*

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**Definition ($\mathcal{T}$-CUD)**

The stock-price $S$ satisfies the **$\mathcal{T}$-conditional Up’n’Down Property ($\mathcal{T}$-CUD)**, if, $\mathbf{P}$-a.s.,

\[
\mathbf{P} \left[ S_{\tau_{j+1}} - S_{\tau_j} > 0 | \mathcal{F}_{\tau_j} \right] > 0, \text{ and } \\
\mathbf{P} \left[ S_{\tau_{j+1}} - S_{\tau_j} < 0 | \mathcal{F}_{\tau_j} \right] > 0,
\]

or $\mathbf{P}[S_{\tau_{j+1}} = S_{\tau_j} | \mathcal{F}_{\tau_j}] = 1$ for all $\tau_j \leq \tau_{j+1}$ satisfying $\mathcal{T}$.

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**Lemma (CUD Characterization of Simple Arbitrage)**

The model $S$ admits $\mathcal{T}$-simple arbitrage if and only if it does not satisfy $\mathcal{T}$-CUD.
**Definition (No Obvious Arbitrage (NOA))**

$S$ satisfies **No Obvious Arbitrage (NOA)** if for all stopping times $\tau$ and $\varepsilon > 0$

$$P \left[ \inf_{t \in [\tau, T]} X_t - X_\tau > -\varepsilon \right] > 0 \text{ and } P \left[ \sup_{t \in [\tau, T]} X_t - X_\tau < \varepsilon \right] > 0$$

**Remark**

If NOA is violated, then there exists 0-admissible **Obvious Arbitrage** of the type $\Phi = \pm 1_{[\tau_1, \tau_2]}$. 
**Simple Arbitrage**
**Infinitesimal Up’n’Down (IUD)**

**Definition (Infinitesimal Up’n’Down (IUD))**

Let $\tau$ be a stopping time. Let

$$\tau^\pm = \inf\{t \geq \tau; \pm S_t > \pm S_{\tau}\}.$$  

$S$ satisfies **Infinitesimal Up’n’Down (IUD)** if $\tau^+ = \tau^- \ P\text{-a.s.}$ for all stopping times.

**Theorem (No Simple arbitrage, NOA and IUD)**

1. $S$ satisfies IUD iff $S$ does not admit 0-admissible simple arbitrage.
2. $S$ satisfies IUD and NOA iff $S$ does not admit simple arbitrage.
Checking if a model satisfies NOA seems very difficult, since it involves stopping times. Ditto for IUD. There is the following though:

**Theorem (Bender 2010)**

Let $S = M + Y$, where $M$ is a continuous martingale with bracket $\langle M \rangle$ and $Y$ is $1/2$-Hölder continuous w.r.t. $\langle M \rangle$. Then $S$ satisfies IUD.

**Sketch of Proof.**

By time-change, consider the Brownian case $M = W$, and use the strong Markov property together with the law of iterated logarithm.
**Outline**

1. **Simple Arbitrage**

2. **Conditional Full Support**

3. **Delay-Simple Arbitrage**
**Definition (Conditional Full Support (CFS))**

$S$ satisfies **Conditional Full Support (CFS)** if, $\mathbb{P}$-a.s.,

$$\text{supp} \left( \text{Law} \left( (S_u)_{u \in [t,T]} \mid \mathcal{F}_t \right) \right) = C^+_S([t,T])$$

**Remark**

- In CFS one can replace $t$ by a stopping time $\tau$!!
- There are host of theorems for checking CFS for given models.
- CFS implies NOA.
- **Unfortunately, CFS does not seem to imply IUD.**
Outline

1. Simple Arbitrage
2. Conditional Full Support
3. Delay-Simple Arbitrage
Since CFS does not (probably) imply IUD, we need to restrict the class of simple strategies. Here the condition $\mathcal{T}$ comes into play.

**Definition (Class $\mathcal{T}_{\text{delay}}$ of Stopping Times)**

For any stopping time $\tau$, let $C_{S_\tau}^+([\tau, T])$ be the random space of continuous positive paths $\omega = (\omega_t)_{t\in[\tau(\omega), T]}$ with $\omega_\tau(\omega) = S_\tau(\omega)(\omega)$ fixed.

A sequence of non-decreasing stopping times $\tau = (\tau_j)_{j=0}^n$ satisfies the **delay** property $\mathcal{T}_{\text{delay}}$ if for all $\tau_k$ there is an $\mathcal{F}_{\tau_j}$-measurable open delay set $U_k \subset C_{S_{\tau_j}}^+([\tau_j, T])$ and an $\mathcal{F}_{\tau_j}$-measurable a.s. positive random variable $\varepsilon_k$ such that $\tau_{j+1} - \tau_j \geq \varepsilon_j$ in the set $U_j \cap \{\tau_{j+1} > \tau_j\}$. 

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Delay-Simple Arbitrage
Examples of Stopping Times with or without Delay

**Example (Positive Example)**

Let $a^j, b^j$ be continuous functions with $a^j_{\tau_j} < 0 < b^j_{\tau_j}$ and let

$$\tau_{j+1} = \inf \left\{ t > \tau_j; S_t - S_{\tau_j} \leq a^j_t \text{ or } S_t - S_{\tau_j} \geq b^j_t \right\}.$$

**Example (Negative Example)**

Let $(\tau_1, \tau_2) = (0, \tau)$ with

$$\tau = \inf \left\{ t > 0; S_t = e^{W_t + t^a} = 1 \right\}$$

for some $a < 1/2$. By the law of iterated logarithm $\tau > 0$, but any open set in $C_1^+([0, T])$ contains sequences $(\omega^n)$ with $\tau(\omega^n) \to 0$. 
Delay-Simple Arbitrage
Delay and Local Continuity

**Definition (Local Continuity (LC))**

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be metric spaces. A function \( f : \mathcal{X} \to \mathcal{Y} \) is **Locally Continuous (LC)** if for all \( x \in \mathcal{X} \) there exists an open \( U_x \subset \mathcal{X} \) such that \( x \in \overline{U_x} \) and \( \lim f(x_n) = f(x) \) whenever \( \lim x_n = x \) in \( U_x \).

The function \( f \) is **Locally Lower-Semicontinuous (LLSC)**, if in the above, \( \lim \inf f(x_n) \geq f(x) \).

**Remark**

LC at \( x \) is continuity from the “direction” \( U_x \). However, LC is not directional continuity in the classical sense. If \( x \in U_x \) then LC is classical continuity.
Example

A functional \( \tau : C_0^+(0, T] \rightarrow [0, T] \) defined by

\[
\tau(\omega) = \min \{ t; \omega(t) = c \}
\]

is LC. Indeed, for \( \omega_0 \in C_0^+([0, T]) \), take

\[
U_{\omega_0} = \{ \omega; \omega(t) > \omega_0(t) \text{ for all } t \in [0, T] \}.
\]

Lemma (Local Lower Semicontinuity and Delay)

*If a stopping time \( \tau \) is LLSC, then it has the delay property.*
**Theorem**

*CFS implies no delay-simple arbitrage.*

**Proof.**

We need to show that the $\mathcal{T}_{\text{delay}}$-CUD is satisfied. We may assume $\tau_{j+1} > \tau_j$. We show that $\mathbb{P}[S_{\tau_{j+1}} > S_{\tau_j}|\mathcal{F}_{\tau_j}] > 0$ a.s.; the proof for $\mathbb{P}[S_{\tau_{j+1}} < S_{\tau_j}|\mathcal{F}_{\tau_j}] > 0$ a.s. follows analogously.

By the CFS it is enough to show that $\{S_{\tau_{j+1}} > S_{\tau_j}\} \subset C_{S_{\tau_j}}^+([\tau_j, T])$ contains an open set. Let $U_j$ be an $\varepsilon_j$-delay set for $\tau_j$.

We first assume that $U_j$ contains a strictly increasing paths $\omega^0$ on $[\tau_j, T]$. Denote by $B_{\omega^0}(\eta_j)$ the open $\eta_j$-ball around $\omega^0$. Choosing $\eta_j$ sufficiently small we have $B_{\omega^0}(\eta_j) \subset U_j$ and $\omega^0_{\tau_j + \varepsilon_j} > \omega^0_{\tau_j} + \eta_j$. 
Hence,

\[ \omega_{\tau_{j+1}}(\omega) - S_{\tau_j} > \omega^0_{\tau_{j+1}}(\omega) - \eta_j - S_{\tau_j} \]
\[ \geq \omega^0_{\tau_j + \epsilon_j} - S_{\tau_j} - \eta_j \]
\[ = \omega^0_{\tau_j + \epsilon_j} - \omega^0_{\tau_j} - \eta_j \]
\[ > 0, \]

So, \( B_{\omega^0}(\eta_j) \subset \{ S_{\tau_{j+1}} > S_{\tau_j} \} \), and the claim follows, if \( U_j \) contains a strictly increasing paths.

If \( U_j \) does not contain a strictly increasing path, we proceed as follows:
Proof, cont., cont.

Being an open set in $C_{S_{\tau_j}}^+ ([\tau_j, T])$, $U_j$ contains paths that are strictly increasing on a small enough interval $[\tau_j, \tau_j + 2\eta_j]$.

Hence, there is a strictly increasing path $\omega^0$ and an open ball $B_j$ around $\omega^0$ in $C_{S_{\tau_j}}^+ ([\tau_j, T])$ such that any $\omega \in B_j$ coincides with some path $\bar{\omega} \in U_j$ on the segment $[\tau_j, \tau_j + \eta_j]$.

Hence, $\tau_{j+1}(\omega) - \tau_j \geq (\tau_{j+1}(\bar{\omega}) - \tau_j) \wedge \eta_j \geq \epsilon_j \wedge \eta_j =: \epsilon^0_j$ for every $\omega \in B_j$.

Therefore $B_j$ is an $\epsilon^0_j$-delay set which contains a strictly increasing path and so the first case applies. $\square$