No-Arbitrage with Non-Semimartingales

Continuous Simple Arbitrage Case

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Abstract

We discuss some recent notions and results connected to simple arbitrage with general continuous pricing models. The discussion is based on the following works (and other works):

- Bender, C. (2010) Simple Arbitrage. *Preprint.*
- Bender, C., Sottinen, T., and Valkeila, E. (2011) Fractional procesess as models in stochastic finance. *Advanced Mathematical Methods for Finance* (Eds. G. Di Nunno and B. Oksendal), Springer.
- Bender, C., Sottinen, T. and Valkeila, E. (2008) Pricing by hedging and no-arbitrage beyond semimartingales. *Finance and Stochastics* **12**, 441–468.

Outline

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$S = (S_t)_{t\in[0,T]}$ is a continuous DISCOUNTED STOCK-PRICE PROCESS on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$.

Definition ($\mathcal{T}$-Simple Strategy)

A $\mathcal{T}$-SIMPLE STRATEGY is a trading strategy where the number of stocks owned by the investor at time t is

$$\Phi_t = \varphi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \varphi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}.$$

φ_j are $\mathcal{F}_{\tau_j}$-measurable, $n \in \mathbb{N}$ is fixed, $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ are stopping times, satisfying some additional condition $\mathcal{T}$ to be specified later. If there are no additional conditions, we omit the $\mathcal{T}$.

The wealth process associated with strategy Φ with initial endowment v is

$$V_t(\Phi, v) = v + \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}}(S_{t\wedge\tau_{j+1}} - S_{t\wedge\tau_j}).$$

Definition ((0-Admissible) $\mathcal{T}$-Simple Arbitrage)

A $\mathcal{T}$-Simple strategy is

- a $\mathcal{T}$-Simple Arbitrage if $V_T(\Phi, 0) \geq 0$ **P**-a.s. and $\mathbf{P}[V_T(\Phi, 0) > 0] > 0$.
- c-Admissible if $V_t(\Phi, 0) \geq -c$ for all $t \in [0, T]$ **P**-a.s.

Definition ($\mathcal{T}$-CUD)

The stock-price S satisfies the $\mathcal{T}$-conditional Up'n'Down Property ($\mathcal{T}$-CUD), if, $\mathbf{P}$-a.s.,

$$\begin{aligned} \mathbf{P}\left[S_{\tau_{j+1}} - S_{\tau_j} > 0 | \mathcal{F}_{\tau_j}\right] &> 0, \text{ and} \\ \mathbf{P}\left[S_{\tau_{j+1}} - S_{\tau_j} < 0 | \mathcal{F}_{\tau_j}\right] &> 0, \end{aligned}$$

or $\mathbf{P}[S_{\tau_{j+1}} = S_{\tau_j} | \mathcal{F}_{\tau_j}] = 1$ for all $\tau_j \leq \tau_{j+1}$ satisfying $\mathcal{T}$.

Lemma (CUD Characterization of Simple Arbitrage)

The model S admits $\mathcal{T}$-simple arbitrage if and only if it does not satisfy $\mathcal{T}$-CUD.

Definition (No Obvious Arbitrage (NOA))

Ssatisfies No Obvious Arbitrage (NOA) if for all stopping times $\tau$and $\varepsilon > 0$

$$\mathbf{P}\left[\inf_{t\in[\tau,T]} X_t - X_\tau > -\varepsilon\right] > 0 \text{ and } \mathbf{P}\left[\sup_{t\in[\tau,T]} X_t - X_\tau < \varepsilon\right] > 0$$

Remark

If NOA is violated, then there exists 0-admissible Obvious Arbitrage of the type $\Phi = \pm\mathbf{1}_{[\tau_1,\tau_2)}$.

Definition (Infinitesimal Up'n'Down (IUD))

Let τ be a stopping time. Let

$$\tau^{\pm} = \inf\{t \geq \tau; \pm S_t > \pm S_\tau\}.$$

S satisfies Infinitesimal Up'n'Down (IUD) if $\tau^+ = \tau^-$ $\mathbf{P}$-a.s. for all stopping times.

Theorem (No Simple arbitrage, NOA and IUD)

1. *S satisfies IUD iff S does not admit 0-admissible simple arbitrage.*
2. *S satisfies IUD and NOA iff S does not admit simple arbitrage.*

Checking if a model satisfies NOA seems very difficult, since it involves stopping times. Ditto for IUD. There is the following though:

Theorem (Bender 2010)

Let $S = M + Y$, where M is a continuous martingale with bracket $\langle M \rangle$ and Y is $1/2$-Hölder continuous w.r.t. $\langle M \rangle$. Then S satisfies IUD.

Sketch of Proof.

By time-change, consider the Brownian case $M = W$, and use the strong Markov property together with the law of iterated logarithm.

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Definition (Conditional Full Support (CFS))

S satisfies Conditional Full Support (CFS) if, **P**-a.s.,

$$\mathrm{supp}\left(\mathrm{Law}\left((S_u)_{u\in[t,T]} \mid \mathcal{F}_t\right)\right) = C^+_{S_t}([t,T])$$

Remark

- In CFS one can replace t by a stopping time τ!!!
- There are host of theorems for checking CFS for given models.
- CFS implies NOA.
- Unfortunately, CFS does not seem to imply IUD.

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Since CFS does not (probably) imply IUD, we need to restrict the class of simple strategies. Here the condition $\mathcal{T}$ comes into play.

Definition (Class $\mathcal{T}_{\text{delay}}$ of Stopping Times)

For any stopping time τ, let $C^+_{S_\tau}([\tau, T])$ be the random space of continuous positive paths $\omega = (\omega_t)_{t\in[\tau(\omega),T]}$ with $\omega_{\tau(\omega)} = S_{\tau(\omega)}(\omega)$ fixed.

A sequence of non-decreasing stopping times $\boldsymbol{\tau} = (\tau_j)_{j=0}^n$ satisfies the delay property $\mathcal{T}_{\text{delay}}$ if for all τ_k there is an $\mathcal{F}_{\tau_j}$-measurable open *delay set* $U_k \subset C^+_{S_{\tau_j}}([\tau_j, T])$ and an $\mathcal{F}_{\tau_j}$-measurable a.s. positive random variable ε_k such that $\tau_{j+1} - \tau_j \geq \varepsilon_j$ in the set $U_j \cap \{\tau_{j+1} > \tau_j\}$.

Example (Positive Example)

Let a^j, b^j be continuous functions with $a^j_{\tau_j} < 0 < b^j_{\tau_j}$ and let

$$\tau_{j+1} = \inf\left\{t > \tau_j; S_t - S_{\tau_j} \leq a^j_t \text{ or } S_t - S_{\tau_j} \geq b^j_t\right\}.$$

Example (Negative Example)

Let $(\tau_1, \tau_2) = (0, \tau)$ with

$$\tau = \inf\left\{t > 0; S_t = e^{W_t + t^a} = 1\right\}$$

for some $a < 1/2$. By the law of iterated logarithm $\tau > 0$, but any open set in $C_1^+([0, T])$ contains sequences (ω^n) with $\tau(\omega^n) \to 0$.

Definition (Local Continuity (LC))

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is Locally Continuous (LC) if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $\lim f(x_n) = f(x)$ whenever $\lim x_n = x$ in U_x.

f is Locally Lower-Semicontinuous (LLSC), if in the above, $\liminf f(x_n) \geq f(x)$.

Remark

LC at x is continuity from the “direction” U_x. However, LC is not directional continuity in the classical sense. If $x \in U_x$ then LC is classical continuity.

Example

A functional $\tau : C^+_{S_0}([0, T]) \to [0, T]$ defined by

$$\tau(\omega) = \min\{t; \omega(t) = c\}$$

is LC. Indeed, for $\omega_0 \in C^+_{S_0}([0, T])$, take

$$U_{\omega_0} = \{\omega; \omega(t) > \omega_0(t) \text{ for all } t \in [0, T]\}.$$

Lemma (Local Lower Semicontinuity and Delay)

If a stopping time τ is LLSC, then it has the delay property.

Theorem

CFS implies no delay-simple arbitrage.

Proof.

We need to show that the $\mathcal{T}_{\text{delay}}$-CUD is satisfied. We may assume $\tau_{j+1} > \tau_j$. We show that $\mathbf{P}[S_{\tau_{j+1}} > S_{\tau_j} | \mathcal{F}_{\tau_j}] > 0$ a.s.; the proof for $\mathbf{P}[S_{\tau_{j+1}} < S_{\tau_j} | \mathcal{F}_{\tau_j}] > 0$ a.s. follows analogously.

By the CFS it is enough to show that $\{S_{\tau_{j+1}} > S_{\tau_j}\} \subset C^+_{S_{\tau_j}}([\tau_j, T])$ contains an open set. Let U_j be an ε_j-delay set for τ_j.

We first assume that U_j contains a strictly increasing paths ω^0 on $[\tau_j, T]$. Denote by $B_{\omega^0}(\eta_j)$ the open η_j-ball around ω^0. Choosing η_j sufficiently small we have $B_{\omega^0}(\eta_j) \subset U_j$ and $\omega^0_{\tau_j + \epsilon_j} > \omega^0_{\tau_j} + \eta_j$.

Proof, cont.

Hence,

$$\begin{aligned}
\omega_{\tau_{j+1}(\omega)} - S_{\tau_j} &> \omega^0_{\tau_{j+1}(\omega)} - \eta_j - S_{\tau_j} \\
&\geq \omega^0_{\tau_j+\varepsilon_j} - S_{\tau_j} - \eta_j \\
&= \omega^0_{\tau_j+\varepsilon_j} - \omega^0_{\tau_j} - \eta_j \\
&> 0,
\end{aligned}$$

So, $B_{\omega^0}(\eta_j) \subset \{S_{\tau_{j+1}} > S_{\tau_j}\}$, and the claim follows, if U_j contains a strictly increasing paths.

If U_j does not contain a strictly increasing path, we proceed as follows:

Proof, cont., cont.

Being an open set in $C^+_{S_{\tau_j}}([\tau_j, T])$, U_j contains paths that are strictly increasing on a small enough interval $[\tau_j, \tau_j + 2\eta_j]$.

Hence, there is a strictly increasing path ω^0 and an open ball B_j around ω^0 in $C^+_{S_{\tau_j}}([\tau_j, T])$ such that any $\omega \in B_j$ coincides with some path $\bar{\omega} \in U_j$ on the segment $[\tau_j, \tau_j + \eta_j]$.

Hence, $\tau_{j+1}(\omega) - \tau_j \geq (\tau_{j+1}(\bar{\omega}) - \tau_j) \wedge \eta_j \geq \epsilon_j \wedge \eta_j =: \epsilon^0_j$ for every $\omega \in B_j$.

Therefore B_j is an ϵ^0_j-delay set which contains a strictly increasing path and so the first case applies. $\square$