Replication and Absence of Arbitrage in Non-Semimartingale Models

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Tommi Sottinen, University of Helsinki (Helsinki)

A joint work with

C. Bender, Weierstrass Institute of Applied Analysis and Stochastics (Berlin)

E. Valkeila, Helsinki University of Technology (Espoo)

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1. Classical arbitrage pricing theory (1/3)

Stock-price process, self-financing strategies, and their wealth

- Discounted market model is $(\Omega, \mathcal{F}, (S_t), (\mathcal{F}_t), \mathbf{P})$. The stock-price process S takes values in $\mathcal{C}_{s_0,+}$ (continuous positive paths on $[0, T]$ starting from s_0).

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- Non-anticipating trading strategy Φ is self-financing if its wealth satisfies

$$V_t(\Phi, v_0; S) = v_0 + \int_0^t \Phi_t \, \mathrm{d}S_t.$$

Here the economic notion 'self-financing' is captured by the 'forward' construction of the Itô integral.

- The strategy Φ is arbitrage (free lunch) if

$$\mathbf{P}\left[V_T(\Phi, 0; S) \geq 0\right] = 1 \quad \text{and} \quad \mathbf{P}\left[V_T(\Phi, 0; S) > 0\right] > 0.$$

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- Efficient market hypothesis: No arbitrage.
- Fundamental theorem of asset pricing: No arbitrage iff S is a semimartingale.
- Option is a mapping $G : \mathcal{C}_{s_0,+} \to \mathbb{R}$. Its fair price is the capital v_0 of a hedging strategy Φ:

$$G(S) = V_T(\Phi, v_0; S).$$

If an option can be hedged then the hedging capital v_0 is unique. Indeed, otherwise there would be arbitrage.

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- There is no arbitrage (S is a semimartingale, fundamental theorem of asset pricing), all options can be hedged, and the hedge is unique (martingale representation theorem).
- Statistically the Black-Scholes model (and more genarally semimartingale models) and the Reality do not seem to agree (stylized facts).

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Even in the classical arbitrage pricing theory one considers only 'admissible' strategies (e.g. doubling strategies have to be ruled out by some ad hoc condition).

We consider a class of pricing models that includes non-semimartingale models. Our aim is to construct a class of 'allowed' strategies for this model class that is

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(i) sufficiently small to exclude arbitrage,
(ii) sufficiently large to contain hedges for relevant option,
(iii) economically meaningful.

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  (a) the classical Black-Scholes model (which we call the reference model $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{S}_t), (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}})$),
  (b) any model of the type
  $$S_t = s_0 e^{\sigma W_t + \frac{\sigma^2}{2} t + Z_t},$$
  Z independent of W, continuous, and satisfies the small ball property. So, we can have heavy tails, long-range dependence, and (almost) any autocorrelation function.

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$$\Phi_t = \varphi\left(t, S_t, S_t^*, S_{*,t}, \bar{S}_t\right),$$

where $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}_+ \times \mathbb{R}^3)$,

$$S_t^* := \max_{r \in [0,t]} S_r, \quad S_{*,t} := \min_{r \in [0,t]} S_r, \quad \bar{S}_t := \int_0^t S_r \, \mathrm{d}r,$$

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(ii) and satisfies the classical 'no doubling strategies' condition

$$\int_0^t \Phi_r \, \mathrm{d}S_r \;\geq\; -a \qquad \mathbf{P} - \text{a.s}$$

for all $t \in [0,T]$ for some $a > 0$.

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We consider non-semimartingales. So, we have no Itô-integrals. However, the forward integral is economically meaningful ((π_n) is a fixed sequence of, say, dyadic partitions of $[0, T]$):

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$$\lim_{n\to\infty} \sum_{\substack{t_k \in \pi_n \\ t_k \le t}} \Phi_{t_{k-1}} \left(S_{t_k} - S_{t_{k-1}} \right).$$

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- Let $u \in \mathcal{C}^{1,2,1}([0, T], \mathbb{R}_+, \mathbb{R}^m)$ and $Y^1, \dots, Y^m$ be bounded variation processes. If S has pathwise quadratic variation (along (π_n)) then we have the Itô formula for $u(t, S_t, Y_t^1, \dots, Y_t^m)$:

$$\mathrm{d}u \;=\; \frac{\partial u}{\partial t}\,\mathrm{d}t + \frac{\partial u}{\partial x}\,\mathrm{d}S + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}\langle S\rangle + \sum_{i=1}^m \frac{\partial u}{\partial y_i}\,\mathrm{d}Y^i.$$

This implies that the forward integral on the right hand side exists and has a continuous modification.

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Idea of Proof: Set, as the Itô formula suggests,

$$
\begin{aligned}
v(t,\eta;\varphi) \quad := \quad & u(t,\eta(t),\eta^*(t),\eta_*(t),\bar{\eta}(t)) \\
& -\int_0^t \frac{\partial u}{\partial t}(r,\eta(r),\eta^*(r),\eta_*(r),\bar{\eta}(r))\,\mathrm{d}r \\
& -\int_0^t \frac{\partial u}{\partial y_1}(r,\eta(r),\eta^*(r),\eta_*(r),\bar{\eta}(r))\,\mathrm{d}\eta^*(r) \\
& -\int_0^t \frac{\partial u}{\partial y_2}(r,\eta(r),\eta^*(r),\eta_*(r),\bar{\eta}(r))\,\mathrm{d}\eta_*(r) \\
& -\int_0^t \frac{\partial u}{\partial y_3}(r,\eta(r),\eta^*(r),\eta_*(r),\bar{\eta}(r))\,\mathrm{d}\bar{\eta}(r) \\
& -\frac{1}{2}\int_0^t \frac{\partial \varphi}{\partial x}(r,\eta(r),\eta^*(r),\eta_*(r),\bar{\eta}(r))\,\sigma^2\eta(r)^2\,\mathrm{d}r,
\end{aligned}
$$

where

$$
u(t,x,y_1,y_2,y_3) \quad = \quad \int_{s_0}^t \varphi(t,\xi,y_1,y_2,y_3)\,\mathrm{d}\xi.
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6. A no-arbitrage result (2/2)

Idea of Proof

Now we have the functional connection

$$V_t(\Phi, v_0; S) \;=\; v_0 + \int_0^t \Phi_t \, \mathrm{d}S_t \;=\; v_0 + v(t, S; \varphi).$$

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Suppose then that $V_T(\Phi, 0; S) = v(T, S; \varphi) \geq 0$ **P**-a.s. By the small ball property and the continuity of $v(t, \cdot; \varphi)$ we have the functional inequality $v(T, \eta; \varphi) \geq 0$ for all $\eta \in \mathcal{C}_{s_0,+}$.

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Now we can go to the reference model and see that $v(T, \tilde{S}; \varphi) \geq 0$ $\tilde{\mathbf{P}}$-a.s. But the classical martingale arguments tell us that then $v(T, \tilde{S}; \varphi) = 0$ $\tilde{\mathbf{P}}$-a.s.

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The claim follows now by interchanging the roles of $\tilde{S}$ and S. □

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Theorem RH Suppose a continuous option $G : \mathcal{C}_{s_0,+} \rightarrow \mathbb{R}$. If $G(\tilde{S})$ can be hedged in the reference model $\tilde{S} \in \mathcal{M}_\sigma$ with an allowed strategy then $G(S)$ can be hedged in any model $S \in \mathcal{M}_\sigma$.

Moreover, the hedges are – as strategies of the stock-path – independent of the model.

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Corollary PDE In the Black-Scholes model hedges for European, Asian, and lookback-options can be constructed by using the Black-Scholes partial differential equation. These hedges hold for any model that is continuous, satisfies the small ball property, and has the same quadratic variation as the Black-Scholes model.

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- Don't use the historical volatility! Instead, use either implied volatility or estimate the quadratic variation (which may be difficult).

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(d) We can relax the smoothness of φ at $t = T$ (this is needed in many classical hedges).

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1. $g(t, \eta) = g(t, \tilde{\eta})$ whenever $\eta(r) = \tilde{\eta}(r)$ on $r \in [0, t]$,
2. $g(\cdot, \eta)$ is of bounded variation and continuous,
3. $$\left| \int_0^t f(u) \mathrm{d}g(u, \eta) - \int_0^t f(u) \mathrm{d}g(u, \tilde{\eta}) \right| \leq K \| f \mathbf{1}_{[0,t]} \|_\infty \cdot \| \eta - \tilde{\eta} \|_\infty$$

(c) The 'strategy functional' φ needs only to be piecewise smooth.

(d) We can relax the smoothness of φ at $t = T$ (this is needed in many classical hedges).

(e) The continuity of the payoff G can be relaxed to include e.g. digital options.

10. References

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