Conditional-Mean Hedging Under Transaction Costs in Gaussian Models

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Workshop on Mathematical Finance, University of Barcelona,
29–30 March 2017

Abstract

We consider invertible Gaussian Volterra processes and derive a formula for their prediction laws. Examples of such processes include the fractional Brownian motions and the mixed fractional Brownian motions.

As an application, we consider conditional-mean hedging under transaction costs in Black-Scholes type pricing models where the Brownian motion is replaced with a more general invertible Gaussian Volterra process.

Outline

1. Invertible Gaussian Volterra Processes and Their Prediction
2. Conditional-Mean Hedging Under Transaction Costs for Geometric Invertible Gaussian Volterra Models
3. Open Problems

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1. Invertible Gaussian Volterra Processes and Their Prediction

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Invertible Gaussian Volterra Processes

Definition (Invertible Gaussian Volterra Process)

A centered continuous Gaussian process X on $[0, T]$ with $X_0 = 0$ is an invertible Gaussian Volterra process if there exists a continuous Gaussian martingale M and kernels k and k^{-1} such that

$$
\begin{aligned}
X_t &= \int_0^t k(t,s)\,\mathrm{d}M_s, \\
M_t &= \int_0^t k^{-1}(t,s)\,\mathrm{d}X_s.
\end{aligned}
$$

As an example, the mixed fractional Brownian motion

$$X_t = aW_t + bB_t, \quad a, b \geq 0,$$

is an invertible Gaussian Volterra process.

Invertible Gaussian Volterra Processes

Let $\xi = (\xi', \xi'')$ be a jointly Gaussian thing. The theorem of Gaussian correlation states that the conditional thing $\xi'|\xi''$ is a Gaussian thing with ξ''-measurable mean and deterministic covariance.

For invertible Gaussian Volterra processes this theorem can be made practical. The key observation is:

$$\mathcal{F}_u^X \quad = \quad \mathcal{F}_u^M \quad =: \quad \mathcal{F}_u.$$

Let $t \geq s \geq u$. Denote

$$\begin{aligned} \hat{X}_t(u) &= \mathbf{E}[X_t|\mathcal{F}_u], \\ \hat{r}(t,s|u) &= \mathbf{Cov}[X_t, X_s, |\mathcal{F}_u]. \end{aligned}$$

Invertible Gaussian Volterra Processes

Now, for the conditional mean, since $\mathcal{F}_u^X = \mathcal{F}_u^M$, we have that

$$\hat{X}_t(u) \;=\; \mathbf{E}[X_t|\mathcal{F}_u] \;=\; \mathbf{E}\left[\int_0^t k(t,s)\,\mathrm{d}M_s\middle|\mathcal{F}_u^M\right].$$

Since Gaussian martingales have independent increments, we obtain

$$\begin{aligned}\hat{X}_t(u) &= \mathbf{E}\left[\int_0^u k(t,s)\,\mathrm{d}M_s + \int_u^t k(t,s)\,\mathrm{d}M_s\middle|\mathcal{F}_u^M\right] \\ &= \int_0^u k(t,s)\,\mathrm{d}M_s.\end{aligned}$$

In order to write $\hat{X}_t(u)$ as a linear transformation of the trajectory X_v, $v \in [0,u]$, we note the following TRANSFER PRINCIPLE.

Invertible Gaussian Volterra Processes

Theorem (Transfer Principle)

$$\int_0^T g(t)\,\mathrm{d}X_t \;=\; \int_0^T K_T^* g(t)\,\mathrm{d}M_t,$$

where K_T^* is the operator defined by linearly extending the relation

$$K_T^* \mathbf{1}_{[0,t)}(s) \;=\; k(t,s).$$

If k is smooth enough and vanishes around the diagonal, then

$$K_T^* g(t) \;=\; \int_t^T g(s)\,k(\mathrm{d}s,t).$$

Invertible Gaussian Volterra Processes

By the (inverse) transfer principle we can write

$$\begin{aligned}
\hat{X}_t(u) &= \int_0^u k(t,s)\,\mathrm{d}M_s \\
&= \int_0^u k(u,s)\,\mathrm{d}M_s - \int_0^u [k(u,s) - k(t,s)]\,\mathrm{d}M_s \\
&= X_u - \int_0^u (K_u^*)^{-1}[k(u,\cdot) - k(t,\cdot)](s)\,\mathrm{d}X_s \\
&=: X_u - \int_0^u \Psi(t,s|u)\,\mathrm{d}X_s.
\end{aligned}$$

Note that if k^{-1} is smooth enough and vanishes around the diagonal, then

$$(K_T^*)^{-1}g(t) = \int_t^T g(s)\,k^{-1}(\mathrm{d}s,t).$$

Invertible Gaussian Volterra Processes

For the conditional covariance

$$\begin{aligned}\hat{r}(t,s|u) &= \mathbf{Cov}[X_t, X_s|\mathcal{F}_u^X] \\ &= \mathbf{E}\left[\left(X_t - \hat{X}_t(u)\right)\left(X_s - \hat{X}_s(u)\right)\Big|\mathcal{F}_u^X\right]\end{aligned}$$

we have

$$\begin{aligned}X_t - \hat{X}_t(u) &= \int_0^t k(t,v)\,\mathrm{d}M_v - \int_0^u k(t,v)\,M_v \\ &= \int_u^t k(t,v)\,\mathrm{d}M_v, \\ X_s - \hat{X}_s(u) &= \int_0^s k(s,w)\,\mathrm{d}M_w - \int_0^u k(s,w)\,M_w \\ &= \int_u^s k(s,w)\,\mathrm{d}M_w.\end{aligned}$$

Invertible Gaussian Volterra Processes

Consequently, since M, being a Gaussian martingale, has independent increments, we have that

$$\begin{aligned}\hat{r}(t,s|u) &= \mathbf{E}\left[\int_u^t k(t,v)\mathrm{d}M_v \int_u^s (s,w)\mathrm{d}M_w \Big| \mathcal{F}_u^M\right] \\ &= \mathbf{E}\left[\int_u^t k(t,v)\mathrm{d}M_v \int_u^s (s,w)\mathrm{d}M_w\right]\end{aligned}$$

Denote

$$q(t) = \mathbf{Var}[M_t].$$

Note that q is also the QUADRATIC VARIATION of M.

Then, by the Itô isometry,

$$\hat{r}(t,s|u) = \int_u^{t\wedge s} k(t,v)k(s,v)\,\mathrm{d}q(v).$$

Invertible Gaussian Volterra Processes

Finally, we note that by the Itô isometry, we have

$$\begin{aligned} r(t,s) &:= \mathbf{Cov}[X_t X_s] \\ &= \mathbf{E}\left[\int_0^t k(t,v)\mathrm{d}M_v \int_0^s k(s,w)\mathrm{d}M_w\right] \\ &= \int_0^{t\wedge s} k(t,v)k(s,v)\,\mathrm{d}q(v). \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{r}(t,s|u) &= \int_u^{t\wedge s} k(t,v)k(s,v)\,\mathrm{d}q(v) \\ &= \int_0^{t\wedge s} k(t,v)k(s,v)\,\mathrm{d}q(v) - \int_0^u k(t,v)k(s,v)\,\mathrm{d}q(v) \\ &= r(t,s) - \int_0^u k(t,v)k(s,v)\,\mathrm{d}q(v). \end{aligned}$$

Invertible Gaussian Volterra Processes

By the theorem of Gaussian correlation, we have the following:

Theorem (Prediction)

The conditional process $X_t|\mathcal{F}_u$, $t \in [u, T]$ *is Gaussian with mean and covariance given by*

$$\begin{aligned} \hat{X}_t(u) &= X_u - \int_0^u \Psi(t, s|u)\,\mathrm{d}X_s, \\ \hat{r}(t, s|u) &= r(t, s) - \int_0^u k(t, v)k(s, v)\,\mathrm{d}q(v). \end{aligned}$$

In what follows, we shall us the following short-hand

$$\hat{r}(t|u) \quad := \quad \hat{r}(t, t|u).$$

Invertible Gaussian Volterra Processes

Some final remarks on the invertible Gaussian Volterra processes:

1. If X is given by the covariance r, then the Volterra representation property means that there exist a measure q and a kernel k such that

$$r(t,s) \;=\; \int_0^{t\wedge s} k(t,v)k(s,v)\,\mathrm{d}q(v).$$

2. The inverse Volterra representation property means that for all $t \in [0,T]$ there exists a function g_t such that

$$K_T^* g_t \;=\; \mathbf{1}_{[0,t)}.$$

3. It is possible that X is continuous, but M is not.

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3 Open Problems

Hedging Under Transaction

We consider (imperfect) hedging under transaction costs in a discounted market model, where the the risky asset is given by

$$\mathrm{d}S_t = S_t\left(\mu(t)\mathrm{d}t + \mathrm{d}X_t\right).$$

Let q be the quadratic variation of the fundamental martingale M associated with X via the kernel k. Then, the quadratic variation of X is (UNDER SOME ASSUMPTIONS)

$$\sigma^2(t)\,\mathrm{d}t := k(t+,t)\,\mathrm{d}q(t).$$

(We also assume that $\sigma^2(t) \not\equiv 0$, and note that that case is even simpler than the one studied here.)

We note that

$$\mathcal{F}_u := \mathcal{F}_u^S = \mathcal{F}_u^X = \mathcal{F}_u^M.$$

Hedging Under Transaction

Let $\pi = (\beta, \gamma)$ be a portfolio. Under continuous hedging without transaction costs its SELF-FINANCING value is

$$\begin{aligned} V_t &= \beta_t + \gamma_t S_t, \\ V_t &= V_0 + \int_0^t \gamma_u \, \mathrm{d}S_u. \end{aligned}$$

Suppose then that there are PROPORTIONAL TRANSACTION COSTS $\kappa \in [0, 1)$ and (consequently) the trading takes place at times t_i, $i = 0, \ldots, n$. Let $\pi^n = (\beta^n, \gamma^n)$ be a portfolio that is rebalanced at times t_i. Then the SELF-FINANCING value of π^n is

$$\begin{aligned} V_t^{n,\kappa} &= \beta_t^n + \gamma_t^n S_t, \\ V_t^{n,\kappa} &= V_0^{n,\kappa} + \int_0^t \gamma_u^n \, \mathrm{d}S_u - \kappa \int_0^t S_u \mathrm{d}|\gamma_u^n|. \end{aligned}$$

Hedging Under Transaction

Let the trading times t_i and the proportional transaction cost κ be fixed. Let $f(S_T)$ be a European vanilla-type option. We are interested in solving the following imperfect hedging or tracking problem.

Definition (Conditional-Mean Hedging)

The strategy π^n is a conditional-mean hedge of the option $f(S_T)$ if for all t_i

$$\mathbf{E}[V^{n,\kappa}_{t_{i+1}}|\mathcal{F}_{t_i}] \;\; = \;\; \mathbf{E}[V_{t_{i+1}}|\mathcal{F}_{t_i}],$$

where V is the value of the continuous-time perfect hedge without transaction costs.

Hedging Under Transaction

Let $\pi = (\beta, \gamma)$ be the replicating strategy of $f(S_T)$ without transaction costs. Then, by the Föllmer–Itô formula

$$\gamma_t = \frac{\partial g}{\partial x}(t, S_t),$$

where $g(t, S_t) = V_t$ comes from the backward PDE

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 g}{\partial x^2} &= 0, \\ g(T,x) &= f(x). \end{aligned}$$

Indeed, the derivation is the same as in the Black–Merton–Scholes case by using Föllmer integration, with the observation

$$(\mathrm{d}S_t)^2 = S_t^2\sigma^2(t)\,\mathrm{d}t.$$

Hedging Under Transaction

Let us then find out the conditional-mean values.

We denote

$$\begin{aligned}\hat{V}_{t_{i+1}}(t_i) &:= \mathbf{E}\left[V_{t_{i+1}} \middle| \mathcal{F}_{t_i}\right], \\ \Delta \hat{V}_{t_{i+1}} &:= \hat{V}_{t_{i+1}}(t_i) - V_{t_i},\end{aligned}$$

$$\begin{aligned}\hat{V}^{n,\kappa}_{t_{i+1}}(t_i) &:= \mathbf{E}\left[V^{n,\kappa}_{t_{i+1}} \middle| \mathcal{F}_{t_i}\right], \\ \Delta \hat{V}^{n,\kappa}_{t_{i+1}} &:= \hat{V}^{n,\kappa}_{t_{i+1}}(t_i) - V^{n,\kappa}_{t_i},\end{aligned}$$

$$\begin{aligned}\hat{S}_{t_{i+1}}(t_i) &:= \mathbf{E}\left[S_{t_{i+1}} \middle| \mathcal{F}_{t_i}\right], \\ \Delta \hat{S}_{t_{i+1}} &:= \hat{S}_{t_{i+1}}(t_i) - S_{t_i},\end{aligned}$$

We will show that all these objects can be calculated explicitly by using the prediction formula for $X_{t_{i+1}}|\mathcal{F}_{t_i}$.

We note that if the initial portfolio $\pi_0^n = (\beta_0^n, \gamma_0^n)$ is fixed, then $V^{n,\kappa}$ can be recovered from the CONDITIONAL-MEAN RECURSION

$$\Delta \hat{V}^{n,\kappa}_{t_{i+1}}(t_i) \quad = \quad \Delta \hat{V}_{t_{i+1}}(t_i).$$

Let us then consider $\hat{V}_{t_{i+1}}(t_i)$:

$$\begin{aligned}
\hat{V}_{t_{i+1}}(t_i) &= \mathbf{E}\left[V_{t_{i+1}} | \mathcal{F}_{t_i}\right] \\
&= \mathbf{E}\left[g(t_{i+1}, S_{t_{i+1}}) \middle| \mathcal{F}_{t_i}\right] \\
&= \mathbf{E}\left[g\left(t_{i+1}, S_0 \mathrm{e}^{\int_0^{t_{i+1}}\left[\mu(v)-\frac{1}{2}\sigma^2(v)\right]\mathrm{d}v + X_{t_{i+1}}}\right) \middle| \mathcal{F}^X_{t_i}\right] \\
&= \mathbf{E}\left[g\left(t_{i+1}, S_0 \mathrm{e}^{\int_0^{t_{i+1}}\left[\mu(v)-\frac{1}{2}\sigma^2(v)\right]\mathrm{d}v + (X_{t_{i+1}} | \mathcal{F}^X_{t_i})}\right)\right].
\end{aligned}$$

Hedging Under Transaction

By using the Gaussian prediction formula, we obtain

$$\begin{aligned}
\hat{V}_{t_{i+1}}(t_i) &= \mathbf{E}\left[g\left(t_{i+1}, S_0 \mathrm{e}^{\int_0^{t_{i+1}}\left[\mu(v)-\frac{1}{2}\sigma^2(v)\right]\mathrm{d}v+(X_{t_{i+1}}|\mathcal{F}_{t_i}^X)}\right)\right] \\
&= \int_{\mathbb{R}} g\left(t_{i+1}, S_0 \mathrm{e}^{\int_0^{t_{i+1}}\left[\mu(v)-\frac{1}{2}\sigma^2(v)\right]\mathrm{d}v+z}\right) \\
&\quad \times\varphi(z;\hat{X}_{t_{i+1}}(t_i), \hat{r}(t_{i+1}|t_i))\mathrm{d}z,
\end{aligned}$$

where $\varphi(\cdot; m, s^2)$ is the density of $\mathrm{N}(m, s^2)$ distribution.

Therefore, both $\hat{V}_{t_{i+1}}(t_i)$ and

$$\begin{aligned}
\Delta\hat{V}_{t_{i+1}}(t_i) &= \int_{\mathbb{R}} g\left(t_{i+1}, S_0 \mathrm{e}^{\int_0^{t_{i+1}}\left[\mu(v)-\frac{1}{2}\sigma^2(v)\right]\mathrm{d}v+z}\right) \\
&\quad \times\varphi(z;\hat{X}_{t_{i+1}}(t_i), \hat{r}(t_{i+1}|t_i))\mathrm{d}z - g\left(t_i, S_{t_i}\right)
\end{aligned}$$

are explicit.

Hedging Under Transaction

Next, we note that

$$
\begin{aligned}
&\hat{S}_{t_{i+1}}(t_i) \\
&\quad = \mathbf{E}\left[S_{t_{i+1}} \middle| \mathcal{F}_{t_i}\right], \\
&\quad = \int_{\mathbb{R}} S_0 \mathrm{e}^{\int_0^{t_{i+1}} \left[\mu(v) - \frac{1}{2}\sigma^2(v) \mathrm{d}v + z\right]} \varphi(z; \hat{X}_{t_{i+1}}, \hat{r}(t_{i+1}|t_i)) \, \mathrm{d}z
\end{aligned}
$$

is explicit.

Consequently,

$$
\begin{aligned}
&\Delta \hat{S}_{t_{i+1}}(t_i) \\
&\quad = \hat{S}_{t_{i+1}}(t_i) - S_{t_i} \\
&\quad = \int_{\mathbb{R}} S_0 \mathrm{e}^{\int_0^{t_{i+1}} \left[\mu(v) - \frac{1}{2}\sigma^2(v) \mathrm{d}v + z\right]} \varphi(z; \hat{X}_{t_{i+1}}, \hat{r}(t_{i+1}|t_i)) \, \mathrm{d}z - S_{t_i}
\end{aligned}
$$

is also explicit.

Hedging Under Transaction

Finally, we note that

$$
\begin{aligned}
&\hat{V}^{n,\kappa}_{t_{i+1}}(t_i) \\
&= \mathbf{E}\left[V^{n,\kappa}_{t_{i+1}} | \mathcal{F}_{t_i}\right] \\
&= V^{n,\kappa}_{t_i} + \mathbf{E}\left[\int_{t_i}^{t_{i+1}} \gamma^n_u \, \mathrm{d}S_u - \kappa \int_{t_i}^{t_{i+1}} S_u \, |\mathrm{d}\gamma^n_u| \Big| \mathcal{F}_{t_i}\right] \\
&= V^{n,\kappa}_{t_i} + \mathbf{E}\left[\gamma^n_{t_i}(S_{t_{i+1}} - S_{t_i}) - \kappa S_{t_i} |\gamma^n_{t_i}| \Big| \mathcal{F}_{t_i}\right] \\
&= V^{n,\kappa}_{t_i} + \mathbf{E}\left[\gamma^n_{t_i} S_{t_{i+1}} \Big| \mathcal{F}_{t_i}\right] - \mathbf{E}\left[\gamma^n_{t_i} S_{t_i} \Big| \mathcal{F}_{t_i}\right] - \mathbf{E}\left[\kappa S_{t_i} |\gamma^n_{t_i}| \Big| \mathcal{F}_{t_i}\right] \\
&= V^{n,\kappa}_{t_i} + \gamma^n_{t_i}\left(\hat{S}_{t_{i+1}}(t_i) - S_{t_i}\right) - \kappa S_{t_i} |\gamma^n_{t_i}|.
\end{aligned}
$$

Consequently,

$$
\Delta \hat{V}^{n,\kappa}_{t_{i+1}}(t_i) = \gamma^n_{t_i} \Delta \hat{S}_{t_{i+1}}(t_i) - \kappa S_{t_i} |\gamma^n_{t_i}|.
$$

Hedging Under Transaction

We note that we have explicit (albeit horrible) formulas for all the quantities we need. Consequently, we have the following result:

Theorem (Conditional-Mean Hedging)

Once the initial portfolio $\pi_0^n = (\beta_0^n, \gamma_0^n)$ is fixed, the conditional-mean hedging portfolio for a European vanilla-type option $f(S_T)$ can be calculated recursively from the explicit formulas for $\Delta \hat{V}_{t_{i+1}}(t_i)$ and $\Delta \hat{S}_{t_{i+1}}(t_i)$ by using the relations

$$
\begin{aligned}
\Delta \hat{V}^{n,\kappa}_{t_{i+1}}(t_i) &= \Delta \hat{V}_{t_{i+1}}(t_i), \\
\Delta \hat{V}^{n,\kappa}_{t_{i+1}}(t_i) &= \gamma^n_{t_i} \Delta \hat{S}_{t_{i+1}}(t_i) - \kappa S_{t_i} |\gamma^n_{t_i}|.
\end{aligned}
$$

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3 Open Problems

Open Problems

1. Which Gaussian processes are invertible Gaussian Volterra processes? If the assumption of continuity is lifted, then my guess is that virtually all.
2. What is the natural initial condition for the conditional-mean hedge? $V_0 = V_0^{n,\kappa}$ or $V_0 = V_0^{n,\kappa} - \kappa S_0|\gamma_0^n|$? Neither of these fixes the initial portfolio $\pi_0^n = (\beta_0^n, \gamma_0^n)$!
3. The conditional-mean hedging is natural from a tracking point of view. One can also consider the minimization problem

$$\mathbf{E}\left[\left(V_T^{n,\kappa} - V_T\right)^2\right] \to \min!$$

This can be "solved" by using the prediction formula. The "solution" is a complete mess. It would be nice to know what is the connection to the conditional-mean hedge.

Thank you for listening!

Any questions?

Any solutions to the open problems?