Conditional Small Balls and No-Arbitrage

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Background

This talk is based on the manuscript

[BSV] Bender, C., Sottinen, T., and Valkeila, E. (2006) *Pricing by hedging and no-arbitrage beyond semimartingales*, 20 p.

So this is ongoing joint work with Christian Bender (TU-Braunschweig) and Esko Valkeila (TKK).

Outline

1. Preliminaries from [BSV]
    - Quadratic variation
    - Model classes
    - Hindsight factors
    - Allowed strategies
    - Robust hedging and no-arbitrage results

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6. Verifying conditional small ball property

Definition

Given a refining sequence of partitions (π_n)

$$\langle S \rangle_t := \lim_{n\to\infty} \sum_{t_i\in\pi_n, t_i\le t} \left(S_{t_i} - S_{t_{i-1}}\right)^2$$

is the pathwise quadratic variation of S (w.r.t. (π_n)).

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Example

For Black-Scholes model (or geometric Brownian motion)

$$dS_t = S_t\mu\, dt + S_t\sigma\, dW_t$$

we have

$$d\langle S \rangle_t = \sigma^2 S_t^2\, dt.$$

Definition

A discounted market model is a 5-tuple $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ where $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space and S is $\mathcal{F}_t$-adapted.

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Definition

Model $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ belongs to model class $\mathcal{M}_{\sigma,s_0}$ if

1. $S_0 = s_0$,
2. $d\langle S\rangle_t = \sigma^2 S_t^2 dt$,
3. for all $\eta : [0, T] \to \mathbb{R}_+$ with $\eta(0) = s_0$ and $\epsilon > 0$

$$\text{(SBP)} \qquad P\left[\sup_{t\in[0,T]} |S_t - \eta(t)| \le \epsilon\right] > 0.$$

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Example

The Black-Scholes model belongs to the model class $\mathcal{M}_{\sigma,s_0}$.

Definition

Mapping $g : [0, T] \times C[0, T] \to \mathbb{R}$ is a hindsight factor if

1. $g(t, \eta) = g(t, \tilde{\eta})$ if $\eta(u) = \tilde{\eta}(u)$ for $u \in [0, t]$,
2. $t \mapsto g(t, \eta)$ is continuous and of bounded variation,
3. for all continuous functions f

$$\left| \int_0^t f(u) dg(u, \eta) - \int_0^t f(u) dg(u, \tilde{\eta}) \right| \\ \leq \quad C \max_{u \in [0,t]} |f(u)| \max_{u \in [0,t]} |\eta(u) - \tilde{\eta}(u)|.$$

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Example

Running maximum, minimum, and average are hindsight factors.

Definition

A trading strategy Φ is allowed if it is admissible for the Black-Scholes model and there exists a smooth φ and hindsight factors $g_1, \ldots, g_m$ such that

$$\Phi_t = \varphi\left(t, S_t, g_1(t, S), \ldots, g_m(t, S)\right).$$

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All models in $\mathcal{M}_{\sigma,s_0}$ are free of arbitrage with allowed strategies.

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If an option can be replicated in one model in $\mathcal{M}_{\sigma,s_0}$ with an allowed strategy then it can be replicated in all models in $\mathcal{M}_{\sigma,s_0}$ with an allowed strategy. Moreover, the replicating strategy is, as a functional of the stock-path, independent of the particular model in $\mathcal{M}_{\sigma,s_0}$.

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- The proofs of the theorems are based on the fact that the wealth associated to an allowed strategy is continuous in the stock path.

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Definition

A simple strategy Φ is of the form

$$\Phi_t = \sum_{i=1}^{n} \Phi^i 1_{(\tau_{i-1},\tau_i]}(t),$$

where Φ_i's are $\mathcal{F}_{\tau_{i-1}}$-measurable and τ_i's are $\mathcal{F}_t$-stopping times.

3. No-arbitrage with take-the-money-and-run

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Lemma

Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ satisfies (SBP). Then for all stopping times τ

$$P[S_\tau > s_0] > 0 \quad \textit{and} \quad P[S_\tau < s_0] > 0.$$

So, take-the-money-and-run strategies $\Phi^0 1_{[0,\tau]}(t)$ are free of arbitrage.

3. No-arbitrage with take-the-money-and-run

Proof.

We only show that $P[S_\tau > s_0] > 0$. The proof for $P[S_\tau < s_0] > 0$ is symmetric.

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Fix an increasing and concave path η_0 with $\eta_0(0) = s_0$.

Since τ is lower semi-continuous we can find such an $\varepsilon < 1/2\,(\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2\,\tau(\eta_0)$ whenever $\eta \in B_{\eta_0}(\varepsilon)$.

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Since η_0 is increasing and concave

$$\begin{aligned} \eta(\tau(\eta)) &> \eta_0(\tau(\eta)) - 1/2\ (\eta_0(\tau(\eta_0)) - s_0) \\ &\geq \eta_0\,(1/2\,\tau(\eta_0)) - 1/2\,\eta_0(\tau(\eta_0)) + 1/2\,s_0 \\ &\geq 1/2\,\eta_0(0) + 1/2\,s_0 = s_0. \end{aligned}$$

□

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Example

Consider the Black-Scholes model with the following twist: Let $\alpha > s_0$ be some level, $\tau = \inf\{t; S_t \geq \alpha\} \wedge T$, and let $T_0 \subset [0, T)$ be some measurable set for which $P[\tau \in T_0] \in (0,1)$. Assume that the stock price follows the Black-Scholes model until τ. Then, if $\tau \in T_0$ the stock price will follow a fixed path η_0 such that $\eta_0(T) > \eta_0(\tau)$. (Of course we assume that η_0 has the correct quadratic variation). If $\tau \notin T_0$, then the stock price will continue to follow the Black-Scholes model.

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Now $1_{\{\tau \in T_0\}} 1_{(\tau, T]}(t)$ is an arbitrage opportunity.

5. No-arbitrage with simple strategies under conditional small ball property

Definition

A market model $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ satisfies conditional small ball property if for all $\mathcal{F}_t$-stopping times τ, all $\varepsilon > 0$ and all positive continuous functions η with $\eta(\tau) = S_\tau$

$$\text{(CSBP)} \qquad P\Big[\sup_{t\in[\tau,T]} |S_t - \eta(t)| \le \epsilon \,\Big|\, \mathcal{F}_\tau\Big] > 0 \qquad P\text{-a.s.}$$

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Theorem

Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ satisfies (CSBP). Then

$$P[S_{\tau_2} > S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0 \quad \text{and} \quad P[S_{\tau_2} < S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0$$

P-almost surely for all $\mathcal{F}_t$-stopping times $\tau_1 < \tau_2$. Consequently simple strategies are free of arbitrage.

5. No-arbitrage with simple strategies under conditional small ball property

Proof.

The proof of the claim

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is similar to the take-the-money-and-run case. One merely replaces the unconditional probabilities with conditional ones.
The freedom of arbitrage for simple strategies

$$\Phi_t = \sum_{i=1}^{n} \Phi^i 1_{(t_{i-1}, t_i]}(t)$$

follows now from the simple fact that in finite-step model a strategy is an arbitrage opportunity if and only if it is an arbitrage opportunity in some single step. □

6. Verifying conditional small ball property

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Proposition

Let X and Y be independent continuous stochastic processes. Suppose X satisfies (CSBP0): For all $\varepsilon > 0$ and $\eta : [0, T] \to \mathbb{R}$ such that $\eta(\tau) = X_\tau$

$$\text{(CSBP0)} \qquad P\Big[\sup_{t\in[\tau,T]} |X_t - \eta(t)| \leq \epsilon \,\Big|\, \mathcal{F}^X_\tau\Big] > 0 \qquad P\text{-}a.s.$$

Then $X + Y$ satisfies (CSBP0) (with $\mathcal{F}^{X+Y}_t$-stopping times, and $\eta(\tau) = X_\tau + Y_\tau$).

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P-almost-surely, where τ is $\mathcal{F}_t^{X,Y}$-stopping time.

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   P-almost-surely, where τ is $\mathcal{F}_t^{X,Y}$-stopping time.

2. Since X and Y are independent we can take the path Y to be a "known parameter" in the conditional probability above. Then the claim follows from the conditional small ball property of X in the ball centered around the path $\eta - Y$. ($\tau(\cdot, Y)$ is $\mathcal{F}_t^X$-stopping time.)

□

Thank you for listening!

Any questions?