Conditional Small Balls and No-Arbitrage

Tommi Sottinen

University of Helsinki and Reykjavík University

May 5th, 2007
This talk is based on the manuscript


So this is ongoing joint work with Christian Bender (TU-Braunschweig) and Esko Valkeila (TKK).
Outline

1 Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2 Shortcomings of [BSV]
3 No-arbitrage with take-the-money-and-run
4 Arbitrage with wait-and-play
5 No-arbitrage with simple strategies under conditional small ball property
6 Verifying conditional small ball property
Outline

1 Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2 Shortcomings of [BSV]
   - No stopping times
1 Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2 Shortcomings of [BSV]
   - No stopping times

3 No-arbitrage with take-the-money-and-run
Outline

1 Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2 Shortcomings of [BSV]
   - No stopping times

3 No-arbitrage with take-the-money-and-run

4 Arbitrage with wait-and-play
Outline

1. Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2. Shortcomings of [BSV]
   - No stopping times

3. No-arbitrage with take-the-money-and-run

4. Arbitrage with wait-and-play

5. No-arbitrage with simple strategies under conditional small ball property
Outline

1. Preliminaries from [BSV]
   - Quadratic variation
   - Model classes
   - Hindsight factors
   - Allowed strategies
   - Robust hedging and no-arbitrage results

2. Shortcomings of [BSV]
   - No stopping times

3. No-arbitrage with take-the-money-and-run

4. Arbitrage with wait-and-play

5. No-arbitrage with simple strategies under conditional small ball property

6. Verifying conditional small ball property
1. Preliminaries from [BSV]
Quadratic variation

Definition

Given a refining sequence of partitions \((\pi_n)\)

\[
\langle S \rangle_t := \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} (S_{t_i} - S_{t_{i-1}})^2
\]

is the pathwise quadratic variation of \(S\) (w.r.t. \((\pi_n)\)).
1. Preliminaries from [BSV]

Quadratic variation

**Definition**

Given a refining sequence of partitions \((\pi_n)\)

\[
\langle S \rangle_t := \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} (S_{t_i} - S_{t_{i-1}})^2
\]

is the pathwise **quadratic variation** of \(S\) (w.r.t. \((\pi_n)\)).

**Example**

For Black-Scholes model (or geometric Brownian motion)

\[
dS_t = S_t \mu \, dt + S_t \sigma \, dW_t
\]

we have

\[
d\langle S \rangle_t = \sigma^2 S_t^2 \, dt.
\]
A discounted market model is a 5-tuple \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) where \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) is a filtered probability space and \(S\) is \(\mathcal{F}_t\)-adapted.
1. Preliminaries from [BSV]

Model classes

Definition

A discounted market model is a 5-tuple \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) where \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) is a filtered probability space and \(S\) is \(\mathcal{F}_t\)-adapted.

Definition

Model \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) belongs to model class \(\mathcal{M}_{\sigma, s_0}\) if

1. \(S_0 = s_0\),
2. \(d\langle S\rangle_t = \sigma^2 S_t^2 dt\),
3. for all \(\eta : [0, T] \rightarrow \mathbb{R}_+\) with \(\eta(0) = s_0\) and \(\epsilon > 0\)

\((SBP)\) \(P\left[\sup_{t \in [0, T]} |S_t - \eta(t)| \leq \epsilon\right] > 0\).

Example

The Black-Scholes model belongs to the model class \(\mathcal{M}_{\sigma, s_0}\).
1. Preliminaries from [BSV]

Model classes

**Definition**

A discounted **market model** is a 5-tuple \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) where \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) is a filtered probability space and \(S\) is \(\mathcal{F}_t\)-adapted.

**Definition**

Model \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) belongs to model class \(\mathcal{M}_{\sigma, s_0}\) if

1. \(S_0 = s_0\),
2. \(d\langle S\rangle_t = \sigma^2 S_t^2 dt\),
3. for all \(\eta : [0, T] \rightarrow \mathbb{R}_+\) with \(\eta(0) = s_0\) and \(\epsilon > 0\)

\((\text{SBP})\) \[P\left[ \sup_{t \in [0, T]} |S_t - \eta(t)| \leq \epsilon \right] > 0.\]

**Example**

The Black-Scholes model belongs to the model class \(\mathcal{M}_{\sigma, s_0}\).
1. Preliminaries from [BSV]

Hindsight factors

Definition

Mapping $g : [0, T] \times C[0, T] \rightarrow \mathbb{R}$ is a hindsight factor if

1. $g(t, \eta) = g(t, \tilde{\eta})$ if $\eta(u) = \tilde{\eta}(u)$ for $u \in [0, t]$,
2. $t \mapsto g(t, \eta)$ is continuous and of bounded variation,
3. for all continuous functions $f$

$$\left| \int_0^t f(u)dg(u, \eta) - \int_0^t f(u)dg(u, \tilde{\eta}) \right| \leq C \max_{u \in [0, t]} |f(u)| \max_{u \in [0, t]} |\eta(u) - \tilde{\eta}(u)|.$$
1. Preliminaries from [BSV]

Hindsight factors

**Definition**

Mapping $g : [0, T] \times C[0, T] \to \mathbb{R}$ is a **hindsight factor** if

1. $g(t, \eta) = g(t, \tilde{\eta})$ if $\eta(u) = \tilde{\eta}(u)$ for $u \in [0, t]$,
2. $t \mapsto g(t, \eta)$ is continuous and of bounded variation,
3. for all continuous functions $f$

$$
\left| \int_0^t f(u) dg(u, \eta) - \int_0^t f(u) dg(u, \tilde{\eta}) \right| \\
\leq C \max_{u \in [0, t]} |f(u)| \max_{u \in [0, t]} |\eta(u) - \tilde{\eta}(u)|.
$$

**Example**

Running maximum, minimum, and average are hindsight factors.
1. Preliminaries from [BSV]

Allowed strategies

Definition

A trading strategy $\Phi$ is allowed if it is admissible for the Black-Scholes model and there exists a smooth $\varphi$ and hindsight factors $g_1, \ldots, g_m$ such that

$$\Phi_t = \varphi(t, S_t, g_1(t, S), \ldots, g_m(t, S)).$$
1. Preliminaries from [BSV]
Robust hedging and no-arbitrage results

**Theorem (No-arbitrage)**

All models in $\mathcal{M}_{\sigma,s_0}$ are free of arbitrage with allowed strategies.
1. Preliminaries from [BSV]
Robust hedging and no-arbitrage results

**Theorem (No-arbitrage)**

*All models in $\mathcal{M}_{\sigma,s_0}$ are free of arbitrage with allowed strategies.*

**Theorem (Robust hedging)**

*If an option can be replicated in one model in $\mathcal{M}_{\sigma,s_0}$ with an allowed strategy then it can be replicated in all models in $\mathcal{M}_{\sigma,s_0}$ with an allowed strategy. Moreover, the replicating strategy is, as a functional of the stock-path, independent of the particular model in $\mathcal{M}_{\sigma,s_0}$.***
1. Preliminaries from [BSV]
Robust hedging and no-arbitrage results

Theorem (No-arbitrage)

All models in $\mathcal{M}_{\sigma, s_0}$ are free of arbitrage with allowed strategies.

Theorem (Robust hedging)

If an option can be replicated in one model in $\mathcal{M}_{\sigma, s_0}$ with an allowed strategy then it can be replicated in all models in $\mathcal{M}_{\sigma, s_0}$ with an allowed strategy. Moreover, the replicating strategy is, as a functional of the stock-path, independent of the particular model in $\mathcal{M}_{\sigma, s_0}$.

- The proofs of the theorems are based on the fact that the wealth associated to an allowed strategy is continuous in the stock path.
2. Shortcomings of [BSV]

- The Robust hedging result is satisfactory: In the Black-Scholes model the hedges for typical options are allowed.
2. Shortcomings of [BSV]

- The Robust hedging result is satisfactory: In the Black-Scholes model the hedges for typical options are allowed.
- The No-arbitrage result is not very satisfactory: The allowed strategies are continuous in the stock path. However, from the economical point of view it is desirable that simple strategies are free of arbitrage.
2. Shortcomings of [BSV]

- The Robust hedging result is satisfactory: In the Black-Scholes model the hedges for typical options are allowed.
- The No-arbitrage result is not very satisfactory: The allowed strategies are continuous in the stock path. However, from the economical point of view it is desirable that simple strategies are free of arbitrage.
- From now on we work on the canonical space $\Omega = C_{s_0, +}[0, T]$, $S_t(\omega) = \eta(t)$, and $\mathcal{F}_t = \mathcal{F}_t^S$. 

Definition

A simple strategy $\Phi$ is of the form $\Phi_t = \sum_{i=1}^n \Phi_i 1_{(\tau_i-1, \tau_i]}(t)$, where $\Phi_i$'s are $\mathcal{F}_{\tau_i-1}$-measurable and $\tau_i$'s are $\mathcal{F}_t$-stopping times.
2. Shortcomings of [BSV]

- The Robust hedging result is satisfactory: In the Black-Scholes model the hedges for typical options are allowed.
- The No-arbitrage result is not very satisfactory: The allowed strategies are continuous in the stock path. However, from the economical point of view it is desirable that simple strategies are free of arbitrage.
- From now on we work on the canonical space $\Omega = \mathbb{C}_{s_0, +[0, T]}$, $S_t(\omega) = \eta(t)$, and $\mathcal{F}_t = \mathcal{F}_t^S$.

**Definition**

A simple strategy $\Phi$ is of the form

$$\Phi_t = \sum_{i=1}^n \Phi_i 1_{(\tau_{i-1}, \tau_i]}(t),$$

where $\Phi_i$’s are $\mathcal{F}_{\tau_{i-1}}$-measurable and $\tau_i$’s are $\mathcal{F}_t$-stopping times.
3. No-arbitrage with take-the-money-and-run

- Stopping times $\eta \mapsto \tau(\eta)$ are typically not continuous.
3. No-arbitrage with take-the-money-and-run

- Stopping times $\eta \mapsto \tau(\eta)$ are typically not continuous.
- Stopping times are typically lower semi-continuous:

$$\liminf_{\eta \to \eta_0} \tau(\eta) \geq \tau(\eta_0).$$

We shall assume this from now on.
3. No-arbitrage with take-the-money-and-run

- Stopping times $\eta \mapsto \tau(\eta)$ are typically not continuous.
- Stopping times are typically lower semi-continuous:

$$\lim \inf_{\eta \to \eta_0} \tau(\eta) \geq \tau(\eta_0).$$

We shall assume this from now on.

**Lemma**

Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P, S)$ satisfies (SBP). Then for all stopping times $\tau$

$$P[S_\tau > s_0] > 0 \quad \text{and} \quad P[S_\tau < s_0] > 0.$$ 

So, take-the-money-and-run strategies $\Phi^0 1_{[0,\tau]}(t)$ are free of arbitrage.
We only show that $P[S_\tau > s_0] > 0$. The proof for $P[S_\tau < s_0] > 0$ is symmetric.
3. No-arbitrage with take-the-money-and-run

Proof.

We only show that $P[S_{\tau} > s_0] > 0$. The proof for $P[S_{\tau} < s_0] > 0$ is symmetric. We show that the set $\{S_{\tau} > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball $B_{\eta_0}(\varepsilon)$. Then the claim will follow from (SBP).
3. No-arbitrage with take-the-money-and-run

**Proof.**

We only show that \( P[S_T > s_0] > 0 \). The proof for \( P[S_T < s_0] > 0 \) is symmetric.

We show that the set \( \{S_T > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\} \) contains a ball \( B_{\eta_0}(\varepsilon) \). Then the claim will follow from (SBP).

Fix an increasing and concave path \( \eta_0 \) with \( \eta_0(0) = s_0 \).
Proof.

We only show that $P[S_\tau > s_0] > 0$. The proof for $P[S_\tau < s_0] > 0$ is symmetric.

We show that the set $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball $B_{\eta_0}(\varepsilon)$. Then the claim will follow from (SBP).

Fix an increasing and concave path $\eta_0$ with $\eta_0(0) = s_0$. Since $\tau$ is lower semi-continuous we can find such an $\varepsilon < 1/2 (\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2 \tau(\eta_0)$ whenever $\eta \in B_{\eta_0}(\varepsilon)$. 

3. No-arbitrage with take-the-money-and-run

Proof.

We only show that $P[S_\tau > s_0] > 0$. The proof for $P[S_\tau < s_0] > 0$ is symmetric.

We show that the set $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball $B_{\eta_0}(\varepsilon)$. Then the claim will follow from (SBP).

Fix an increasing and concave path $\eta_0$ with $\eta_0(0) = s_0$. Since $\tau$ is lower semi-continuous we can find such an $\varepsilon < 1/2 \ (\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2 \tau(\eta_0)$ whenever $\eta \in B_{\eta_0}(\varepsilon)$.

Since $\eta_0$ is increasing and concave

$$\eta(\tau(\eta)) > \eta_0(\tau(\eta)) - 1/2 \ (\eta_0(\tau(\eta_0)) - s_0)$$

$$\geq \eta_0 \ (1/2 \tau(\eta_0)) - 1/2 \eta_0(\tau(\eta_0)) + 1/2 \ s_0$$

$$\geq \ 1/2 \eta_0(0) + 1/2 \ s_0 = s_0.$$
4. Arbitrage with wait-and-play

- The no-arbitrage result for take-the-money-and-run strategies is unfortunately not enough to ensure no-arbitrage for simple strategies:
4. Arbitrage with wait-and-play

- The no-arbitrage result for take-the-money-and-run strategies is unfortunately not enough to ensure no-arbitrage for simple strategies:

**Example**

Consider the Black-Scholes model with the following twist: Let \( \alpha > s_0 \) be some level, \( \tau = \inf\{t; S_t \geq \alpha\} \wedge T \), and let \( T_0 \subset [0, T) \) be some measurable set for which \( P[\tau \in T_0] \in (0, 1) \). Assume that the stock price follows the Black-Scholes model until \( \tau \). Then, if \( \tau \in T_0 \) the stock price will follow a fixed path \( \eta_0 \) such that \( \eta_0(T) > \eta_0(\tau) \). (Of course we assume that \( \eta_0 \) has the correct quadratic variation). If \( \tau \notin T_0 \), then the stock price will continue to follow the Black-Scholes model.
4. Arbitrage with wait-and-play

- The no-arbitrage result for take-the-money-and-run strategies is unfortunately not enough to ensure no-arbitrage for simple strategies:

**Example**

Consider the Black-Scholes model with the following twist: Let $\alpha > s_0$ be some level, $\tau = \inf\{t; S_t \geq \alpha\} \wedge T$, and let $T_0 \subset [0, T)$ be some measurable set for which $P[\tau \in T_0] \in (0, 1)$. Assume that the stock price follows the Black-Scholes model until $\tau$. Then, if $\tau \in T_0$ the stock price will follow a fixed path $\eta_0$ such that $\eta_0(T) > \eta_0(\tau)$. (Of course we assume that $\eta_0$ has the correct quadratic variation). If $\tau \notin T_0$, then the stock price will continue to follow the Black-Scholes model.

Now $1_{\{\tau \in T_0\}} 1_{(\tau, T]}(t)$ is an arbitrage opportunity.
5. No-arbitrage with simple strategies under conditional small ball property

**Definition**

A market model \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) satisfies **conditional small ball property** if for all \(\mathcal{F}_t\)-stopping times \(\tau\), all \(\varepsilon > 0\) and all positive continuous functions \(\eta\) with \(\eta(\tau) = S_\tau\)

\[
\text{(CSBP)} \quad P\left[ \sup_{t \in [\tau, T]} |S_t - \eta(t)| \leq \varepsilon \middle| \mathcal{F}_\tau \right] > 0 \quad P\text{-a.s.}
\]
5. No-arbitrage with simple strategies under conditional small ball property

**Definition**
A market model \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) satisfies **conditional small ball property** if for all \(\mathcal{F}_t\)-stopping times \(\tau\), all \(\varepsilon > 0\) and all positive continuous functions \(\eta\) with \(\eta(\tau) = S_\tau\)

\[
(\text{CSBP}) \quad P\left[ \sup_{t \in [\tau, T]} |S_t - \eta(t)| \leq \varepsilon \right| \mathcal{F}_\tau > 0 \quad P-a.s.
\]

**Theorem**
Suppose \((\Omega, \mathcal{F}, \mathcal{F}_t, P, S)\) satisfies (CSBP). Then

\[
P[S_{\tau_2} > S_{\tau_1}|\mathcal{F}_{\tau_1}] > 0 \quad \text{and} \quad P[S_{\tau_2} < S_{\tau_1}|\mathcal{F}_{\tau_1}] > 0
\]

\(P\)-almost surely for all \(\mathcal{F}_t\)-stopping times \(\tau_1 < \tau_2\). Consequently simple strategies are free of arbitrage.
5. No-arbitrage with simple strategies under conditional small ball property

Proof.

The proof of the claim

\[ P[S_{\tau_2} > S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0 \quad \text{and} \quad P[S_{\tau_2} < S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0 \]

is similar to the take-the-money-and-run case. One merely replaces the unconditional probabilities with conditional ones.
5. No-arbitrage with simple strategies under conditional small ball property

Proof.

The proof of the claim

\[ P[S_{\tau_2} > S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0 \quad \text{and} \quad P[S_{\tau_2} < S_{\tau_1} | \mathcal{F}_{\tau_1}] > 0 \]

is similar to the take-the-money-and-run case. One merely replaces the unconditional probabilities with conditional ones. The freedom of arbitrage for simple strategies

\[ \Phi_t = \sum_{i=1}^{n} \Phi^i 1_{(t_{i-1}, t_i]}(t) \]

follows now from the simple fact that in finite-step model a strategy is an arbitrage opportunity if and only if it is an arbitrage opportunity in some single step.
6. Verifying conditional small ball property

- The following proposition can be used to verify (CSBP).
6. Verifying conditional small ball property

- The following proposition can be used to verify (CSBP).

**Proposition**

Let $X$ and $Y$ be independent continuous stochastic processes. Suppose $X$ satisfies (CSBP0): For all $\varepsilon > 0$ and $\eta : [0, T] \to \mathbb{R}$ such that $\eta(\tau) = X_\tau$,

$$(CSBP0) \quad P \left[ \sup_{t \in [\tau, T]} |X_t - \eta(t)| \leq \varepsilon \mid \mathcal{F}_\tau^X \right] > 0 \quad P-a.s.$$

Then $X + Y$ satisfies (CSBP0) (with $\mathcal{F}_t^{X+Y}$-stopping times, and $\eta(\tau) = X_\tau + Y_\tau$).
6. Verifying conditional small ball property

Proof.

The proof is based on two ideas:

1. Conditional expectation is a strictly positive operator. Hence, it is enough to show that
   \[ P \left[ \sup_{t \in [\tau, T]} |X_t + Y_t - \eta(t)| \leq \epsilon \right] > 0 \]
   \( P \)-almost-surely, where \( \tau \) is \( F_{X_t, Y_t} \)-stopping time.

2. Since \( X \) and \( Y \) are independent we can take the path \( Y \) to be a "known parameter" in the conditional probability above. Then the claim follows from the conditional small ball property of \( X \) in the ball centered around the path \( \eta - Y \).

\( \tau(\cdot, Y) \) is \( F_{X_t} \)-stopping time.)
6. Verifying conditional small ball property

Proof.

The proof is based on two ideas:

1. Conditional expectation is a strictly positive operator. Hence, it is enough to show that

\[ P \left[ \sup_{t \in [\tau, T]} |X_t + Y_t - \eta(t)| \leq \epsilon \mid \mathcal{F}^{X,Y}_{\tau} \right. \cup \left. \mathcal{F}^{Y}_{T} \right] > 0 \]

\( P \)-almost-surely, where \( \tau \) is \( \mathcal{F}^{X,Y}_{t} \)-stopping time.

Since \( X \) and \( Y \) are independent we can take the path \( Y \) to be a "known parameter" in the conditional probability above. Then the claim follows from the conditional small ball property of \( X \) in the ball centered around the path \( \eta - Y \). (\( \tau(\cdot, Y) \) is \( \mathcal{F}^{X,Y}_t \)-stopping time.)
6. Verifying conditional small ball property

Proof.

The proof is based on two ideas:

1. Conditional expectation is a strictly positive operator. Hence, it is enough to show that

\[ P \left[ \sup_{t \in [\tau, T]} |X_t + Y_t - \eta(t)| \leq \epsilon \mid \mathcal{F}_{\tau}^{X,Y} \vee \mathcal{F}_T^Y \right] > 0 \]

\( P \)-almost-surely, where \( \tau \) is \( \mathcal{F}_t^{X,Y} \)-stopping time.

2. Since \( X \) and \( Y \) are independent we can take the path \( Y \) to be a “known parameter” in the conditional probability above. Then the claim follows from the conditional small ball property of \( X \) in the ball centered around the path \( \eta - Y \).

(\( \tau(\cdot, Y) \) is \( \mathcal{F}_t^X \)-stopping time.)
Thank you for listening!

Any questions?