A celebration of Dynkin's formula

Probabilistic interpretations for ODE's

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A tutorial at
Hunan First Normal University, Changsha, PRC
on
Thursday 26 November, 2015

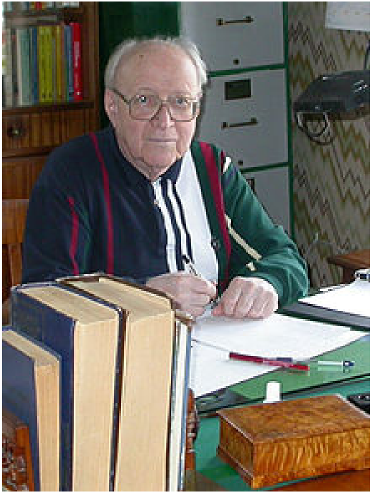
Eugene Dynkin
11 May 1924, Leningrad, USSR — 11 November 2014, Ithaca, USA

Yevgenij "Eugene" Borisovich Dynkin was a Soviet–American mathematician. He was a rare example of a mathematician who made fundamental contributions to two very distinct areas: algebra and probability theory.

Dynkin was one of the founders of the modern theory of Markov processes.

The Dynkin diagram, the Dynkin system, and Dynkin's formula are named for him.

The **Dynkin's formula** builds a bridge between **differential equations** and **Markov processes**.



Eugene Dynkin (1924–2014)

Outline

Outline

The ODE to be solved

Consider the 2^{nd} order ordinary differential equation (ODE)

$$\mu(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x) = 0, \quad x \in (a,b) \subset \mathbb{R},$$

with boundary conditions $u(a)$ and $u(b)$, and parameter functions $\mu(x)$ and $\frac{1}{2}\sigma^2(x)$, given; $u(x)$ is the function to be solved.

The probabilistic approach we use solve the ODE will not only give a way to simulate the solution, but also provides a physical interpretation of the ODE.

Actually, the probabilistic method will easily extend to partial differential equations (PDE's) of even more general nature.

Outline

The Markov property of a stochastic process $X = (X_t)_{t \geq 0}$ states that "the past is independent of the future, given the present".

Thus, a time-homogeneous Markov process is characterized by its transition probabilities

$$p_t(x, \mathrm{d}y) = \mathbf{P}^x[X_t \in \mathrm{d}y] = \mathbf{P}[X_t \in \mathrm{d}y \,|\, X_0 = x],$$

and by the Chapman–Kolmogorov equations

$$p_{t+s}(x, \mathrm{d}y) = \int_{\xi \in \mathbb{R}} p_t(x, \mathrm{d}\xi) p_s(\xi, \mathrm{d}y).$$

Instead of transition probabilities, consider the TRANSITION SEMIGROUP

$$P_t f(x) = \mathbf{E}^x[f(X_t)] = \mathbf{E}[f(X_t) \mid X_0 = x].$$

Then the CHAPMAN–KOLMOGOROV EQUATIONS translate into the SEMIGROUP PROPERTY

$$P_{t+s} = P_t P_s$$

Then, the semigroup P_t has a GENERATOR A, such that

$$P_t = e^{tA}.$$

Of course, here

$$Af(x) = P'_{0+} f(x) = \lim_{t \to 0+} \frac{P_t f(x) - P_0 f(x)}{t}.$$

Outline

A martingale is a stochastic process $M = (M_t)_{t \geq 0}$, where "the best prediction for the future is the present":

$$\mathbf{E}[M_{t+s} \mid M_u, u \leq t] = \mathbf{E}^{M_t}[M_{t+s}] = M_t.$$

Another characterization of martingales is via stopping times.

An X-stopping time is a random time τ such that observing the stochastic process X on the time interval upto time t, you know whether $\tau \leq t$, or not.

A stochastic process M is a martingale if and only if

$$\mathbf{E}[M_\tau] = M_0$$

for all M-stopping times τ.

Martingales (we assume that all our stochastic processes are continuous) have very erratic paths. Indeed the paths are typically nowhere differentiable.

It is a deep result in martingale theory, that martingales M admit quadratic variation:

$$\langle M \rangle_t = \lim_{n\to\infty} \sum_{t_k \in \Pi_n[0,t]} \left(X_{t_k} - X_{t_{k-1}} \right)^2 .$$

The existence of (non-trivial) quadratic variation makes stochastic calculus different from classical calculus.

We will use the informal, but suggestive notation

$$(\mathrm{d}M)^2 = \mathrm{d}\langle M \rangle .$$

Outline

Brownian motion

The Brownian motion (or Wiener process) W is the most important stochastic process in all science.

The Brownian motion can be characterized as the unique Markov process with generator

$$Af(x) = \frac{1}{2} f''(x).$$

$\Longrightarrow$

- Gaussian:

$$p_t(x, \mathrm{d}y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-y)^2}{t}} \,\mathrm{d}y.$$

- stationary, independent increments; martingale
- $(\mathrm{d}W)^2 = \mathrm{d}t.$

Outline

Let M be a martingale and let F be a process such that F_t is observable from $M_s, s \leq t$. The stochastic integral

$$Y_t = \int_0^t F_s \,\mathrm{d}M_s$$

is defined by linearly extending and closing in $L^2(\mathbb{R}_+ \times \Omega)$ the relation

$$\int_{t_1}^{t_2} F_{t_1} \,\mathrm{d}M_s = F_{t_1} \left(M_{t_2} - M_{t_1} \right).$$

Y is a martingale with

$$(\mathrm{d}Y)^2 = F^2 \,(\mathrm{d}M)^2.$$

An Ito diffusion is the solution of the SDE

$$\mathrm{d}X = \mu(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W.$$

The solution can be simulated easily by using the Euler–Maruyama method.

X is a semi-martingale (a martingale + differentiable drift) with

$$(\mathrm{d}X)^2 = \sigma^2(X)\,\mathrm{d}t.$$

X is a Markov process with generator

$$Lf(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Conversely, a Markov process with generator L is the Ito diffusion X.

Outline

THEOREM (ITO'S FORMULA)

Let $g \in C^2$ and let X be a semi-martingale. Then

$$\mathrm{d}g(X) = g'(X)\mathrm{d}X + \frac{1}{2}g''(X)(\mathrm{d}X)^2.$$

Proof: Use the Taylor's formula

$$\begin{aligned} g(X_{t_k}) - g(X_{t_{k-1}}) &= g'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) \\ &\quad + \frac{1}{2}g''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 \\ &\quad + \varepsilon(g'')(X_{t_k} - X_{t_{k-1}})^2. \end{aligned}$$

□

Outline

The geometric Brownian motion is the solution of the geometric SDE

$$\frac{\mathrm{d}X}{X} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W.$$

By Ito's formula the solution is

$$X = X_0 e^{\mu t + \sigma W - \frac{1}{2}\sigma^2 t}.$$

The geometric Brownian motion is important in finance; financial assets, e.g. stocks, are modeled by using it.

The Ornstein–Uhlenbeck process is the solution to the linear SDE

$$\mathrm{d}X = \theta(\mu - X)\,\mathrm{d}t + \sigma\,\mathrm{d}W$$

By Ito's formula the solution is

$$X = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s}\,\mathrm{d}W.$$

The Ornstein–Uhlenbeck process is important in particle Physics; it describes the velocity of a particle under friction. Also, it describes the position of a string under thermal fluctuations.

It is also important in finance; FX rates, e.g. EUR/CNY, are modeled by using it.

Outline

Dynkin's formula

Theorem (Dynkin's formula)

Let $f \in C^2$. Let X be Ito diffusion with generator

$$Lf(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

Then the process

$$df(X) - Lf(X)\,\mathrm{d}t$$

is a martingale.

Proof: Use the Ito's formula. $\square$.

Outline

Probabilistic solution to the ODE to be solved

Solution as expectation at exit time

Recall, we want to solve

$$\mu(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x) \;=\; 0, \quad x \in (a,b) \subset \mathbb{R},$$

which boundary values $u(a)$ and $u(b)$ given.

Let X be the Ito diffusion

$$\mathrm{d}X = \mu(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W.$$

It follows from the Dynkin's formula that

$$u(x) = \mathbf{E}^x[u(X_\tau)],$$

where τ is the first exit time of X from the interval (a,b).

Probabilistic solution to the ODE to be solved

Simulate N independent trajectories X^n, $n = 1, \dots, N$, of the Ito diffusion (by using the Euler–Maruyama method) starting from point x. Let τ^n be the time the trajectory X^n leaves the domain (a, b).

Then approximative solution to the ODE is

$$\hat{u}_N(x) = \frac{1}{N} \sum_{n=1}^{N} u(X^n_{\tau^n}).$$

Outline

The full glory of Dynkin's formula

Dirichlet problem with a 2^{nd} order PDE

Consider the 2^{nd} order PDE operator

$$Af(\mathbf{x}) \;=\; \frac{1}{2}\sum_{i,j=1}^{n}\sigma_{ij}^{(2)}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}f(\mathbf{x}) + \sum_{i=1}^{n}\mu_i(\mathbf{x})\frac{\partial}{\partial x_i}f(\mathbf{x}) + w(\mathbf{x})f(\mathbf{x}).$$

Here $\sigma^{(2)}(\mathbf{x})$ is the square of some $\sigma(\mathbf{x})$:

$$\sigma_{ij}^{(2)}(\mathbf{x}) \;=\; \sum_{k=1}^{n}\sigma_{ik}(\mathbf{x})\,\sigma_{kj}(\mathbf{x}).$$

Let $D \subset \mathbb{R}^d$ be a domain. Consider the PDE boundary value problem

$$\begin{cases} Au(\mathbf{x}) = 0, & \mathbf{x} \in D, \\ u(\mathbf{y}) = f(\mathbf{y}), & \mathbf{y} \in \partial D \,\text{(given)}. \end{cases}$$

The full glory of Dynkin's formula

Probabilistic solution with physical interpretation

Let $\mathbf{X}$ be the d-dimensional Ito diffusion moving in D according to the dynamics

$$\mathrm{d}\mathbf{X} = \mu(\mathbf{X})\mathrm{d}t + \sigma(\mathbf{X})\,\mathrm{d}\mathbf{W}.$$

Let τ be the exit time of $\mathbf{X}$ from D. Then the solution is

$$u(\mathbf{x}) = \mathbf{E}^x\left[e^{\int_0^\tau w(\mathbf{X})\mathrm{d}t} f(\mathbf{X}_\tau)\right].$$

Physical interpretation: Consider a particle system in a stationary state. Small particles (think of molecules and heat) $\mathbf{X}$ move randomly in D. They gain/lose weight with rate $w(\mathbf{x})$, have drift $\mu(\mathbf{x})$, and volatility $\sigma(\mathbf{x})$; $f(\mathbf{y})$ is the observed total weight of the particles that exit at $\mathbf{y} \in \partial D$; $u(\mathbf{x})$ is the number of particles that originate from $\mathbf{x}$.

Thank you for listening! Any questions?